DYNAMICAL TRAJECTORIES: THE MOTION OF A PARTICLE IN AN ARBITRARY FIELD OF FORCE*

BY

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The object of the present paper is to work out for space of three dimensions a geometric theory analogous to that given for the two-dimensional case in an earlier paper. †

We consider a particle moving freely in space under the action of any positional force, and write its equations of motion in the form

$$(1) \qquad \ddot{x} = \phi(x, y, z), \qquad \ddot{y} = \psi(x, y, z), \qquad \ddot{z} = \chi(x, y, z),$$

where the dots indicate derivatives with respect to the time. The functions ϕ, ψ, χ are assumed to have partial derivatives of first and second order in the region of space considered. The case where the force vanishes everywhere is excluded.

The motion of the particle is determined when its initial position and initial velocity are given. By taking all possible initial conditions we obtain a definite quintuply infinite system of trajectories. Our object is to study the properties of such systems of curves with a view to obtaining a *complete geometric characterization*. The main result is stated in article 39 at the end of the paper.

The first properties derived are consequences of the elementary fact that the osculating plane at any point of a trajectory is determined by its initial direction and by the direction of the force (articles 4–11). The next set relate to osculating spheres.‡ With each lineal element there are associated ∞^1 osculating spheres whose centers lie on a straight line (articles 12–15). The straight lines corresponding to all the elements at a point form a congruence of order one and class three determined by a twisted cubic curve (articles 16–23). The properties obtained at this stage belong to a more general class of curve systems than the dynamical class. The final characterization is attained by introducing certain related systems of plane curves, here termed S-systems, which have all the

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† *The Trajectories of Dynamics*, Transactions of the American Mathematical Society, vol. 7 (1906), pp. 401–424. We shall refer to this as *Trajectories*.

‡ Properties I_1, I_2, and II of the present paper were given in a note published in the Bulletin of the American Mathematical Society, vol. 12 (1905), pp. 71–74.

properties of plane systems of trajectories (articles 29–39). The S-system in a given plane may be obtained dynamically by constructing, at each point of the plane, the component of the force (1) acting in that plane; the two-dimensional field thus derived generates the required system.

It will be seen that the spatial properties are not direct extensions of those given in the plane theory; and that the present theory requires the previous discussion of the two-dimensional case. The earlier theory is thus not absorbed by the present paper.

The logical relations of the various properties are settled by converting the main theorems, thus bringing to light general systems of curves of interest apart from their connections with dynamics. Of the incidental results we mention only the interpretation of the curl of the force given in article 21, and the special characterization of conservative forces in articles 22 and 40.*

Differential equations of the trajectories

1. In order to derive the geometrical properties of the trajectories, it is necessary to eliminate the time. We distinguish the two types of derivatives involved by means of dots and primes. Thus

$$\dot{x} = \frac{dx}{dt}, \qquad \dot{y} = \frac{dy}{dt}, \qquad \dot{z} = \frac{dz}{dt}, \text{ etc.},$$

$$y' = \frac{dy}{dx}, \qquad z' = \frac{dz}{dx}, \text{ etc.} \tag{2}$$

The geometric derivatives may be expressed in terms of the kinematic as follows:

$$y' = \frac{\dot{y}}{\dot{x}}, \qquad z' = \frac{\dot{z}}{\dot{x}}, \qquad y'' = \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{\dot{x}^3}, \qquad z'' = \frac{\dot{x}\ddot{z} - \dot{z}\ddot{x}}{\dot{x}^3}. \tag{3}$$

Combining these equations with (1), we obtain two expressions for $\dot{x}^2$, namely

$$\dot{x}^2 = \frac{\psi - y'\phi}{y''}, \qquad \dot{x}^2 = \frac{\chi - z'\phi}{z''}. \tag{4}$$

Thus one result free from the time is

$$\frac{\psi - y'\phi}{y''} = \frac{\chi - z'\phi}{z''}. \tag{5}$$

Differentiating equations (4) with respect to x, we find two other equations

*The transformation theory given at the conclusion of *Trajectories* (articles 26–29) may be extended to space, but is not considered in the present paper. In the discussion of the APPELL transformation (article 30), the formulas should be expressed, not in terms of the times t and t_1, but, as in APPELL's treatment, in terms of the differentials dt and dt_1. The finite representation of page 423 yields merely the affine transformation. The appropriate converse question was solved by PAINLEVÉ, Journal de Mathématiques (1894).

free from the time,

$$(6)\qquad (\psi - y'\phi)y''' = \begin{vmatrix} 1 & \phi_x + y'\phi_y + z'\phi_z \\ y' & \psi_x + y'\psi_y + z'\psi_z \end{vmatrix} y'' - 3\phi y''^2,$$

$$(7)\qquad (\chi - z'\phi)z''' = \begin{vmatrix} 1 & \phi_x + y'\phi_y + z'\phi_z \\ z' & \chi_x + y'\chi_y + z'\chi_z \end{vmatrix} z'' - 3\phi z''^2,$$

where the subscripts denote partial derivatives.

The quintuply infinite system of trajectories is completely represented by the equations (5) *and* (6). We write these in the form

$$(8)\qquad \begin{aligned} z'' &= Ky'', \\ y''' &= Py'' + Qy''^2, \end{aligned}$$

where

$$K = \frac{\chi - z'\phi}{\psi - y'\phi}, \qquad Q = \frac{-3\phi}{\psi - y'\phi},$$

$$(8')\qquad P = \frac{\begin{vmatrix} 1 & \phi_x + y'\phi_y + z'\phi_z \\ y' & \psi_x + y'\psi_y + z'\psi_z \end{vmatrix}}{\psi - y'\phi}.$$

All other equations between the geometric derivatives, including (7), are indeed consequences of (8). This may be seen from the fact that a trajectory is completely determined by the assignment of the initial values of x, y, z, y', z', y''; for this is equivalent, according to (4), to the assignment of initial position and initial velocity. The corresponding values of z'', y''' are given by (8), and those of z''', y^{iv}, z^{iv}, etc., are obtained by successive differentiation of (8).

2. It will be convenient, in part of the subsequent discussion, to define the force whose components are ϕ, ψ, χ by the functions

$$(9)\qquad \omega_1 = \frac{\psi}{\phi}, \qquad \omega_2 = \frac{\chi}{\phi}, \qquad \Phi = \log \phi.$$

The first two of these define the direction of the force; the equations of the lines of force are, in fact,

$$(10)\qquad dx : dy : dz = 1 : \omega_1 : \omega_2.$$

The third depends upon the intensity of the force, and completely defines it when the other two are known.

The coefficients in the fundamental equations (8) may now be written

$$(8'')\qquad \begin{aligned} K &= \frac{z' - \omega_2}{y' - \omega_1}, \qquad Q = \frac{3}{y' - \omega_1}, \\ P &= \Phi_x + y'\Phi_y + z'\Phi_2 - \frac{\omega_{1x} + y'\omega_{1y} + z'\omega_{1z}}{y' - \omega_1}. \end{aligned}$$

By differentiating the first equation (8) and combining with the second, we obtain

$$z''' = Ky''' + K'y'' = \left(P + \frac{K'}{K}\right)z'' + \frac{Q}{K}z''.$$

This may be written in the form

$$z''' = Pz'' + Qz'^2, \tag{10}$$

where

$$P = \Phi_x + y'\Phi_y + z'\Phi_z - \frac{\omega_{2x} + y'\omega_{2y} + z'\omega_{2z}}{z' - \omega_2}, \qquad Q = \frac{3}{z' - \omega_2}, \tag{10'}$$

and is, of course, equivalent to (7).

3. It may be shown, as in *Trajectories*, that if two fields ϕ, ψ, χ and ϕ_1, ψ_1, χ_1, lead to the same trajectories, that is, the same equations (8), then they can differ only by a constant factor. The system of trajectories thus determines the field.*

Osculating planes.

4. *Property* I. The osculating plane of a trajectory at a given point is determined by its initial direction and by the direction of the force acting at the given point. This familiar fact is easily verified analytically by noting that the general equation of the osculating plane is

$$\begin{vmatrix} X & Y & Z \\ 1 & y' & z' \\ 0 & y'' & z'' \end{vmatrix} = 0, \tag{11}$$

where the current coördinates X, Y, Z refer to axes passing through the given point. This is satisfied, in virtue of (5), by $X : Y : Z = \phi : \psi : \chi$.

Property I. — *The osculating planes of all trajectories passing through a given point form a pencil.*

The axis of the pencil has the direction of the force acting at the given point.

5. *Converse of* I. It is easy to determine all quintuply infinite systems of curves possessing property I. Let the direction, fixed for each point, be defined by the ratios

$$1 : \omega_1 : \omega_2,$$

where ω_1 and ω_2 are arbitrary functions of x, y, z. By hypothesis, the osculating plane (11) passes through this direction for all possible values y', z', y'', z''. Hence

$$\begin{vmatrix} 1 & \omega_1 & \omega_2 \\ 1 & y' & z' \\ 0 & y'' & z'' \end{vmatrix} = 0,$$

* A geometric construction of the field will appear in a later paper, *The inverse problem of dynamics.*

which gives

$$\frac{z''}{y''} = \frac{z' - \omega_2}{y' - \omega_1}.$$

Conversely, when this relation is fulfilled, the curves have the required property. Hence:

All quintuply infinite systems of curves possessing property I *are represented by differential equations of the form*

$$z'' = \frac{z' - \omega_2}{y' - \omega_1} y'' = Ky'', \qquad y''' = f(x, y, z, y', z', y''). \tag{12}$$

The result thus contains two arbitrary functions ω_1, ω_2 of x, y, z and an arbitrary function f of the six arguments indicated.

6. *Property* I_1. Before proceeding to essentially new properties, we derive a few corollaries of interest. If the initial position and direction, fixed by the values of x, y, z, y', z', be given, then y'' is arbitrary and we thus have ∞^1 trajectories. For these we may state the elementary result:

The ∞^1 *trajectories corresponding to a given lineal element have (at the given point) a common osculating plane.*

7. *Property* I_2. The torsion of any curve is given by the general formula

$$\frac{1}{\rho} = \frac{y''z''' - z''y'''}{(y'y'' + z'z'')^2 - (1 + y'^2 + z'^2)(y''^2 + z''^2)}. \tag{13}$$

For trajectories, we have

$$z'' = Ky''.$$

Hence

$$z''' = Ky''' + K'y'',$$

where

$$K' = \frac{\omega_1'(z' - \omega_2) - \omega_2'(y' - \omega_1)}{(y' - \omega_1)^2}.$$

Introducing these values in (13), we find

$$\frac{1}{\rho} = \frac{K'}{(y' + Kz')^2 - (1 + y'^2 + z'^2)(1 + K^2)}. \tag{14}$$

This does not involve y'' or z''. Hence the result:

The ∞^1 *trajectories determined by a given lineal element have (at the given point) the same torsion.*

8. *Converse of* I_1 *and* I_2. These properties hold not only for dynamical systems, but for the more general systems (12). They are thus consequences of I and might in fact have been derived synthetically from it. The converse,

however, does not hold. There are in fact systems possessing both I_1 and I_2 (for all lineal elements) but not possessing I. We omit the discussion and state the result:

A system of space curves possesses property I_1 *when the relation between* z'' *and* y'' *is of the form*

$$z'' = k(x, y, z, y', z')y'', \tag{15}$$

where k *is arbitrary. It also possesses property* I_2 *provided the function* k *satisfies the equation*

$$k_{y'} + kk_{z'} = 0. \tag{16}$$

The function K, appearing in (8) and (12), is seen to be a special solution of (16).

8′. *Related helices.* At each point of a space curve it is possible to construct a unique helix having in common with the given curve its direction, its osculating plane, its curvature, and its torsion.* We term this the *related helix*. If the current coördinates X, Y, Z refer to axes determined by the tangent, principal normal, and binormal at the given point, then the axis of the helix is given by the formulas

$$Y = \frac{r\rho^2}{r^2 + \rho^2}, \qquad \frac{X}{Z} = -\frac{r}{\rho}, \tag{17}$$

where r and ρ are the radii of curvature and torsion respectively.

Consider now the ∞^1 trajectories passing through a given point in a given direction. These have a common osculating plane and a common torsion. The curvature varies from curve to curve. The elimination of r from (17) gives

$$Y(X^2 + Z^2) + \rho XZ = 0. \tag{18}$$

This is recognized as PLÜCKER's normal equation for a cylindroid.† Hence:

The ∞^1 *trajectories determined by a given lineal element give rise to related helices whose axes generate a cylindroid.*

The discussion shows that this holds for the more general systems obtained in article 8. The result is thus a consequence of I_1 and I_2.

9. *Distribution of torsion.* In article 7 it was shown that for each lineal element there is a definite value of ρ. If we consider the ∞^2 elements at a given point, they will subdivide into sets of ∞^1 according to the values of the parameter ρ. If ρ has a given value, say c, the corresponding elements are

* SCHEFFERS, *Theorie der Curven*, p. 197. The contact is not of the third order unless dr/ds vanishes. Only in this case does an *osculating helix* exist.

† A cylindroid presents itself in a different connection in SCHEFFER's discussion (l. c., p. 196). There r is fixed, while ρ varies. The elimination of ρ from (17) leads to $Y(X^2+Z^2)-rZ^2=0$.

found, from (14), to be

$$(20)\quad (\omega_1 z' - \omega_2 y')^2 + (y' - \omega_1)^2 + (z' - \omega_2)^2 - c \begin{vmatrix} y' - \omega_1 & \omega_{1x} + y'\omega_{1y} + z'\omega_{1z} \\ z' + \omega_2 & \omega_{2x} + y'\omega_{2y} + z'\omega_{2z} \end{vmatrix}.$$

The elements thus determine the cone

$$(20')\quad (\omega_1 Z - \omega_2 Y)^2 + (Y - \omega_1 X)^2 + (Z - \omega_2 X)^2 - c \begin{vmatrix} Y - \omega_1 X & \omega_{1x} X + \omega_{1y} Y + \omega_{1z} Z \\ Z - \omega_2 X & \omega_{2x} X + \omega_{2y} Y + \omega_{2z} Z \end{vmatrix}.$$

The trajectories which pass through a given point and have there the same torsion are arranged so that their initial directions form a quadric cone.

10. When the given torsion is zero, (20′) becomes

$$(21)\quad \begin{vmatrix} Y - \omega_1 X & \omega_{1x} X + \omega_{1y} Y + \omega_{1z} Z \\ Z - \omega_2 X & \omega_{2x} X + \omega_{2y} Y + \omega_{2z} Z \end{vmatrix} = 0.$$

In this case the osculating plane has four, instead of three, consecutive points in common with the curve.

Of the ∞^3 trajectories through a given point there are ∞^2 with hyperosculating planes. The lineal elements of these trajectories generate the quadric cone (21).

11. By varying c in (20′) we obtain a pencil of cones. It may be shown that all pass through the element defined by the ratios

$$(22)\quad 1 : \omega_1 : \omega_2,$$

and that along this element they have a common tangent plane

$$(23)\quad \begin{vmatrix} 1 & \omega_1 & \omega_2 \\ 0 & \omega_{1x} + \omega_1\omega_{1y} + \omega_2\omega_{1z} & \omega_{2x} + \omega_1\omega_{2y} + \omega_2\omega_{2z} \\ X & Y & Z \end{vmatrix} = 0.$$

The element (22) has the direction of the line of force and the plane (23) is the osculating plane of the line of force.

Osculating spheres.

12. The properties thus far obtained are consequences of property I. To obtain independent properties we now consider osculating spheres. If we take the given point as origin and let X, Y, Z denote the coördinates of the center of the sphere, the equation of the sphere is

$$x^2 + y^2 + z^2 - 2Xx - 2Yy - 2Zz = 0.$$

The conditions for osculation with a given curve are

$$(24)\quad \begin{aligned} X + y'Y + z'Z &= 0, \\ y''Y + z''Z &= 1 + y'^2 + z'^2, \\ y'''Y + z'''Z &= 3(y'y'' + z'z''). \end{aligned}$$

From these we obtain the general formulas

$$X = -\frac{(1+y'^2+z'^2)(y'z''' - z'y''') - 3(y'z'' - z'y'')(y'y'' + z'z'')}{y''z''' - z''y'''},$$

$$(25)\quad Y = \frac{(1+y'^2+z'^2)z''' - 3z''(y'y'' + z'z'')}{y''z''' - z''y'''},$$

$$Z = -\frac{(1+y'^2+z'^2)y''' - 3y''(y'y'' + z'z'')}{y''z''' - z''y'''},$$

13. *Property* II. Consider the ∞^1 trajectories determined by a given lineal element. For these y'' is arbitrary and z'', y''', z''' are determined as functions of y'' by the fundamental differential equations (8) and the derived equation (10). If we substitute these values in (25) or, as is more convenient, in (24), we may eliminate the parameter y'' and obtain two relations between X, Y, Z. One of these is already given by the first equation (24); the other is found to be

$$(26)\qquad PY + KPZ + Q(1+y'^2+z'^2) - 3(y' + Kz') = 0.$$

The coefficients in both these equations are constants since they depend only upon the given lineal element. Since the equations are linear in X, Y, Z, we have

PROPERTY II. *The osculating spheres of the ∞^1 trajectories passing through a given point in a given direction have their centers on a straight line.*

The straight line is necessarily in the plane perpendicular to the given direction at the given point, i. e., in the common normal plane of the ∞^1 trajectories. Since the spheres all pass through the given point, it follows from the above that they pass through a common circle. Hence property II may be restated in this form:

The osculating spheres of the ∞^1 trajectories described above form a pencil.

14. *Converse of* II. It is clear that II is independent of I. In fact the proof of I depended upon only the first of the fundamental equations (8), while that of II involved also the second.

We now find all systems of curves possessing I and II. By article 5 the differential equations are necessarily of the form (12). It remains to determine the form of the function f. The hypothesis is that for each lineal element the locus of the centers (X, Y, Z) of the osculating spheres shall be a straight line. Since the centers are necessarily in the plane normal to the element, the equations of the straight line may be taken in of the form

$$X + y'Y + z'Z = 0,$$

$$A_1X + A_2Y + A_3Z + A_4 = 0,$$

where the coefficients A involve x, y, z, y', z' in any way. Substituting the general values (25) in the second of these equations, we find a relation of the form

$$(1+y'^2+z'^2)(B_1z'''-B_2y''')-3(B_1z''-B_2y'')(y'y''+z'z'')+B_3(y''z'''-z''y''')=0,$$

where the B's involve only x, y, z, y', z'. Introducing the values of z'', y''' given in (12) and the value of z''' found by differentiation, we obtain

$$f = Gy'' + Hy''^2,$$

where G and H are functions of x, y, z, y', z'.

It is easy to verify that this condition is sufficient, that is, that property II holds for all values of G and H. Hence:

The most general quintuply infinite system of space curves possessing properties I *and* II *is defined by equations of the form*

$$z'' = Ky'' = \frac{z'-\omega_2}{y'-\omega_1}y'', \qquad y''' = Gy'' + Hy''^2; \tag{27}$$

the result thus involves two arbitrary functions ω_1, ω_2 *of* x, y, z, *and two arbitrary functions* G, H *of* x, y, z, y', z'.

15. *Correspondence of centers.* We now prove a property which holds not only for dynamical systems but for the general systems just obtained.

For each of the ∞^1 curves corresponding to a given lineal element we may construct a circle of curvature and an osculating sphere. Denote their centers by O' and O'' respectively. By property II the locus of O'' is a straight line. The locus of O' is evidently also a straight line, namely, the principal normal which, by property I_1, is common to all the curves. We now prove that *the ranges described by* O' *and* O'' *are similar.*

Take the given point as origin and the given initial direction as that of the axis of abscissas. Then the distance from O' to the given point, that is, the radius of curvature, is found to be

$$r = \frac{1}{y''\sqrt{1+K^2}}. \tag{28}$$

The ordinate of O'' is

$$Z = \frac{G+Hy''}{K'y''}. \tag{29}$$

Eliminating y'', we obtain a relation of the form

$$Z = ar + b,$$

which proves the result stated. In the dynamical case the property may be expressed as follows:

If a particle is projected from a given point in a given direction, and if for each of the ∞^1 trajectories obtained by varying the initial velocity we construct the center O' of the circle of curvature and the center O'' of the osculating sphere, then the points O' and O'' describe similar ranges.

There is no difficulty in proving this synthetically as a consequence of I and II.

THE CONGRUENCE Ω AND THE CUBIC Γ.

16. *Equations of* Ω. We have seen that tc each lineal element there corresponds a definite straight line, the locus of the centers of the osculating spheres. If the point is kept fixed and the direction of the element varied, then ∞^2 of these straight lines are obtained. What is the character of the congruence Ω thus generated?

The equations of the straight line corresponding to a given element, found in article 13, may be written

$$X + y'Y + z'Z = 0,$$

$$(30)\quad \begin{vmatrix} 1 & \phi_x + y'\phi_y + z'\phi_z \\ y' & \psi_x + y'\psi_y + z'\psi_z \end{vmatrix} Y + \begin{vmatrix} 1 & \phi_x + y'\phi_y + z'\phi_z \\ z' & \chi_x + y'\chi_y + z'\chi_z \end{vmatrix} Z = 3(\phi + y'\psi + z'\chi).$$

Here x, y, z have fixed values and y', z' are arbitrary parameters leading to the ∞^2 straight lines of Ω. The quadratic terms in y', z' may be eliminated by combining the equations, giving the result:

The congruence Ω is defined by the equations

$$(31)\quad X + y'Y + z'Z = 0, \qquad AX + BY + CZ = 3D,$$

where

$$(31')\quad \begin{aligned} A &= y'\phi_y + z'\phi_z, & B &= \psi_x + y'(\psi_y - \phi_x) + z'\psi_z, \\ C &= \chi_x + y'\chi_y + z'(\chi_z - \phi_x), & D &= \phi + y'\psi + z'\chi. \end{aligned}$$

17. *The cubic curve* Γ. If the values of X, Y, Z are given, then, since the equations (31) are linear in the parameters, they will determine, in general, unique values of y', z'. Hence *the congruence is of order one.*

The equations (31) may be written in the form

$$(32)\quad X + Yy' + Zz' = 0, \qquad \alpha + \beta y' + \gamma z' = 0,$$

where

$$\begin{aligned} \alpha &= \psi_x Y + \chi_x Z - 3\phi, \\ \beta &= \phi_y X + (\psi_y - \phi_x) Y + \chi_y Z - 3\psi, \\ \gamma &= \phi_z X + \psi_z Y + (\chi_z - \phi_x) Z - 3\chi. \end{aligned}$$

Hence the element corresponding to a general point X, Y, Z is given by the formulas

$$(33)\quad dx : dy : dz = \Delta_1 : \Delta_2 : \Delta_3,$$

where

$$\Delta_1 = Y\gamma - Z\beta = \phi_z XY - \phi_y XZ + \psi_z Y^2 + (\chi_z - \psi_y) YZ - 3\chi Y - \chi_y Z^2 + 3\psi Z,$$

$$\Delta_2 = Z\alpha - X\gamma = -\phi_z X^2 - \psi_z XY + (\phi_z - \chi_z) XZ + 3\chi X + \psi_x YZ + \chi_x Z^2 + 3\phi Z,$$

$$\Delta_3 = X\beta - Y\alpha = \phi_y X^2 + (\psi_y - \phi_x) XY + \chi_y XZ - 3\psi X - \psi_x Y^2 - \chi_x YZ + 3\phi Y.$$

The point X, Y, Z will be singular if the equations (33) do not determine the element. This will be the case when

$$\Delta_1 = \Delta_2 = \Delta_3 = 0. \tag{34}$$

From the identity

$$X\Delta_1 + Y\Delta_2 + Z\Delta_3 \equiv 0, \tag{35}$$

which is readily verified, it is seen that (34) represents a cubic curve.

The singular points of the congruence Ω *form a twisted cubic curve.*

It will be convenient to have the equations of this curve Γ in parametric form. These may be found by expressing the fact that for a singular point the equations (32) differ only by a factor λ. Hence the required parametric equations are

$$\begin{aligned} \lambda X + \psi_x Y + \chi_x Z &= 3\phi, \\ \phi_y X + (\psi_y - \phi_x + \lambda) Y + \chi_y Z &= 3\psi, \\ \phi_z X + \psi_z Y + (\chi_z - \phi_x + \lambda) Z &= 3\chi. \end{aligned} \tag{36}$$

The explicit expressions for X, Y, Z as cubic functions of the parameter λ will not be needed in the following discussion.

18. *Property* III_1. We now may prove that the congruence Ω is completely defined by the curve Γ, by showing that every straight line of Ω has two singular points, that is, is a secant of Γ.

Since any element may be made to take the direction of the axis of abscissas by revolving the axes, it will be sufficient to prove the statement for the single element $y' = 0$, $z' = 0$. The corresponding straight line, by (31), is

$$X = 0, \qquad \psi_x Y + \chi_x Z - 3\phi = 0.$$

It is easy to verify, by using either (34) or (36), that this line has two points in common with Γ.

PROPERTY III_1. *The straight lines which correspond, according to property* II, *to the* ∞^2 *elements at a given point, form a congruence which is composed of the secants of a twisted cubic curve.*

19. *Property* III_2. The preceding article associates with any given point a

definite cubic Γ. From the equations (34) it is seen that the curve passes through the given point, which is there taken as the origin of the X, Y, Z coördinates. In the parametric representation (36), the origin is given by the value $\lambda = \infty$. The direction of the curve at that point is found to be given by the ratios $\phi : \psi : \chi$; it is therefore the same as the direction of the force acting at the given point. This direction is defined geometrically in connection with property I. We may, therefore, state the new property as follows:

PROPERTY III_2. *The cubic curve* Γ, *associated by property* III_1 *with any point, passes through that point in the direction of the axis of the pencil of osculating planes described in property* I.

20. *Quadrics through* Γ. Consider the ∞^2 trajectories passing through a given point and having initial directions in a given plane. The lineal elements thus satisfy a linear relation

$$a\,dx + b\,dy + c\,dz = 0. \tag{37}$$

It follows, from (33), that the corresponding centres X, Y, Z satisfy the equation

$$a\Delta_1 + b\Delta_2 + c\Delta_3 = 0. \tag{38}$$

The locus of the centers of the osculating spheres of the ∞^2 *trajectories touching a given plane at a given point is a quadric surface.*

The quadrics (38) are in fact the linear system of quadrics passing through the cubic curve Γ. The result is really a corollary of the previous properties and may be derived synthetically. The straight line corresponding to a given element is obtained by constructing the plane perpendicular to that element at the given point; this plane cuts Γ in two new points; the line connecting these is the required line. If then we take the pencil of elements determined by a plane through the given point, the corresponding normal planes form a pencil; hence the corresponding straight lines form a regulus. This determines a quadric of the linear system (38).

21. *Interpretation of the curl.* To each plane (37) through the given point there corresponds a definite quadric (38). For which planes will this quadric be a rectangular hyperboloid?

The condition for such a quadric is that the sum of the coefficients of X^2, Y^2, Z^2 shall vanish. This gives

$$a(\psi_z - \chi_y) + b(\chi_x - \phi_z) + c(\phi_y - \psi_x) = 0. \tag{39}$$

This means that the plane (37) must contain the fixed direction defined by the ratios

$$\psi_z - \chi_y : \chi_x - \phi_z : \phi_y - \psi_x. \tag{40}$$

These three quantities are the components of the curl of the force ϕ, ψ, χ. Hence:

The direction of the curl of the force at any given point is defined geometrically by the fact that the quadrics corresponding, according to article 20, *to the planes containing this direction are rectangular hyperboloids. This is not the case for any other planes through the given point.*

22. *Conservative force.* The force is conservative when, and only when, the curl vanishes identically. In this case condition (39) holds for all values of a, b, c. Hence the quadric corresponding to any plane is a rectangular hyperboloid. The cubic curve Γ is then of a particular species which may also be termed rectangular. Its definitive property may be expressed in simple form: *the three asymptotic directions of the cubic are mutually orthogonal.*

Our result is thus a purely geometric characterization of the conservative case:

When, and only when, the force is conservative are the cubic curves Γ, *corresponding to all points of space in accordance with property* III_1, *of the rectangular species.*

23. *Converse of* III. Our next problem is to determine all curve systems possessing properties III_1 and III_2 in addition to I and II. It will be convenient to refer to the joint statement of III_1 and III_2 as PROPERTY III. The discussion will show incidentally that III is independent of I and II.

The most general system for which I and II hold was proved in article 14 to be represented by equations of the form (27), namely,

$$z'' = Ky'', \qquad y''' = Gy'' + Hy''^2. \tag{41}$$

The straight line corresponding to a given element is here determined by the equations*

$$X + y'Y + z'Z = 0, \qquad MY + NZ + 1 = 0, \tag{42}$$

where

$$M, N = \frac{G, KG + K'}{H(1 + y'^2 + z'^2) - 3(y' + Kz')}. \tag{42'}$$

The problem is to determine the form of the functions G and H, or, what is equivalent, of the functions M and N, so that property III shall also hold, that is, so that the congruence (42) shall be of the particular type described in III_1 and III_2.

Let the equations of the cubic associated with the given point, which we take

* In the dynamical case these reduce, of course, to equations (30).

as origin of the homogeneous cartesian coördinates X, Y, Z, T, be, in parametric form,

$$\begin{aligned} X &= a_4\lambda^3 + a_0\lambda^2 + a_1\lambda + a_2, \\ Y &= b_4\lambda^3 + b_0\lambda^2 + b_1\lambda + b_2, \\ Z &= c_4\lambda^3 + c_0\lambda^2 + c_1\lambda + c_2, \\ T &= d_4\lambda^3 + d_0\lambda^2 + d_1\lambda + d_2. \end{aligned} \tag{43}$$

The curve is to pass through the origin by III_2. We may take the parameter so that $\lambda = \infty$ shall correspond to this point. Therefore

$$a_4 = b_4 = c_4 = 0. \tag{44}$$

The directional property involved in III_2 requires further that

$$a_0 : b_0 : c_0 = 1 : \omega_1 : \omega_2. \tag{45}$$

We may therefore take our cubic curve in the form

$$\begin{aligned} X &= a_0\lambda^2 + a_1\lambda + a_2, \\ Y &= b_0\lambda^2 + b_1\lambda + b_2, \\ Z &= c_0\lambda^2 + c_1\lambda + c_2, \\ T &= \lambda^3 + d_0\lambda^2 + d_1\lambda + d_2. \end{aligned} \qquad (a_0 : b_0 : c_0 = 1 : \omega_1 : \omega_2). \tag{46}$$

The coefficients a, b, c, d may involve x, y, z in any way.

If we substitute these values in (42), we find

$$L_0\lambda^2 + L_1\lambda + L_2 = 0, \tag{47}$$

where

$$L_0 = a_0 + y'b_0 + z'c_0, \quad L_1 = a_1 + y'b_1 + z'c_1, \quad L_2 = a_2 + y'b_2 + z'c_2; \tag{47'}$$

and

$$\lambda^3 + (Mb_0 + Nc_0 + d_0)\lambda^2 + (Mb_1 + Nc_1 + d_1)\lambda + (Mb_2 + Nc_2 + d_2) = 0. \tag{48}$$

The condition that the line (42) shall intersect the cubic (46) in two points is that the equations (47), (48) shall have two roots λ in common. This gives two relations

$$\begin{aligned} L_0(b_0L_2)M + L_0(c_0L_2)N &= L_1L_2 - L_0(d_0L_2), \\ L_0(b_1L_2)M + L_0(c_1L_2)N &= L_2^2 - L_0(d_1L_2), \end{aligned} \tag{49}$$

where the parentheses represent determinants

$$(b_0L_2) = b_0L_2 - b_2L_0, \text{ etc.} \tag{49'}$$

in terms of their principal diagonals.

The solution of (49) gives, after some reduction,

$$M = \frac{L_1(c_1 L_2) - L_2(c_0 L_2) - L_0(d_1 c_2 L_3)}{(abc) L_0},$$

$$N = \frac{L_2(b_0 L_2) - L_1(b_1 L_2) + L_0(d_1 b_2 L_3)}{(abc) L_0}. \tag{50}$$

Arranging these with respect to y', z', we find that the functions M and N are necessarily of the form

$$M = \frac{m_1 + m_2 y' + m_3 z' + y'(l_1 y' + l_2 z')}{3(1 + \omega_1 y' + \omega_2 z')},$$

$$N = \frac{n_1 + n_2 y' + n_3 z' + z'(l_1 y' + l_2 z')}{3(1 + \omega_1 y' + \omega_2 z')}, \tag{51}$$

where the coefficients l, m, n involve only x, y, z.

Conversely, any functions of this form lead to a system (41) with the required properties. The congruence (42) may, in fact, for these values of M and N, be put into the form

$$X + y' Y + z' Z = 0,$$

$$-(l_1 y' + l_2 z') X + (m_1 + m_2 y' + m_3 z') Y + (n_1 + n_2 y' + n_3 z') Z = 3(1 + \omega_1 y' + \omega_2 z'). \tag{52}$$

A discussion entirely analogous to that given in articles 17 and 18 shows that the congruence is defined by the cubic curve

$$\frac{m_1 Y + n_1 Z + 3}{X} = \frac{m_2 Y + n_2 Z - l_1 X + 3\omega_1}{Y} = \frac{m_3 Y + n_3 Z - l_2 X + 3\omega_2}{Z}. \tag{53}$$

This passes through the origin in the direction $1 : \omega_1 : \omega_2$, hence property III holds in its entirety. Our result may be stated as follows:

The most general quintuply infinite system of curves with properties I, II *and* III *is represented by equations of the form*

$$z'' = Ky'', \qquad y''' = Gy'' + Hy''^2, \tag{54}$$

where

$$K = \frac{z' - \omega_2}{y' - \omega_1}, \tag{54'}$$

and G and H are defined by the relations

$$G : KG + K' : H(1 + y'^2 + z'^2) - 3(y' + Kz') = m_1 + m_2 y' + m_3 z' + y'(l_1 y' + l_2 z') : n_1 + n_2 y' + n_3 z' + z'(l_1 y' + l_2 z') : 3(1 + \omega_1 y' + \omega_2 z'). \tag{54''}$$

The system thus involves ten arbitrary functions of x, y, z, namely,

$$\omega_1,\ \omega_2,\ l_1,\ l_2,\ m_1,\ m_2,\ m_3,\ n_1,\ n_2,\ n_3. \tag{55}$$

Relations between the ten coefficients.

24. The dynamical systems, defined by equations of the form (8) in connection with (8′) or (8″), involve only three arbitrary functions ϕ, ψ, χ and thus constitute a special case of the systems obtained in the preceding article. The question then arises, what is the special nature of the ten functions (55), which appear as coefficients in the equations (54″), if the system (54) is to be of the dynamical type?

Comparing the equations (52) with the corresponding equations (31), (31′) of the dynamical case, and introducing the functions ω_1, ω_2, Φ defined by (9), we find that

$$
\begin{aligned}
& l_1 = \Phi_y, && l_2 = \Phi_z, \\
(56)\quad & m_1 = -\omega_1\Phi_x - \omega_{1x}, && m_2 = \Phi_x - \omega_{1y} - \omega_1\Phi_y, && m_3 = -\omega_{1z} - \omega_1\Phi_z, \\
& n_1 = -\omega_2\Phi_x - \omega_{2x}, && n_2 = -\omega_{2y} - \omega_2\Phi_y, && n_3 = \Phi_x - \omega_{2z} - \omega_2\Phi_z.
\end{aligned}
$$

In order that the ten functions (55) shall belong to a dynamical system it is necessary and sufficient that they shall be expressible in terms of three functions ω_1, ω_2, Φ according to (56).

25. The explicit conditions may be obtained by eliminating Φ from (56). The algebraic elimination of Φ_x, Φ_y, Φ_z yields five relations, namely:

$$
\begin{gathered}
m_3 + \omega_{1z} + \omega_1 l_2 = 0, \qquad n_2 + \omega_{2y} + \omega_2 l_1 = 0, \\
(57)\qquad m_2 + \omega_{1y} + \omega_1 l_1 = n_3 + \omega_{2z} + \omega_2 l_2 = -\frac{m_1 + \omega_{1x}}{\omega_1} = -\frac{n_1 + \omega_{2x}}{\omega_2}.
\end{gathered}
$$

Furthermore, from (56) we obtain the equations

$$
(58)\qquad \Phi_x = -\frac{m_1 + \omega_{1x}}{\omega_1} = -\frac{n_1 + \omega_{2x}}{\omega_2}, \qquad \Phi_y = l_1, \qquad \Phi_z = l_2,
$$

for which the conditions of integrability may be written

$$
(59)\qquad l_{1z} - l_{2y} = 0, \qquad l_{2x} + \left(\frac{n_1 + \omega_{2x}}{\omega_2}\right)_z = 0, \qquad l_{1x} + \left(\frac{m_1 + \omega_{1x}}{\omega_1}\right)_y = 0.
$$

26. We have thus obtained eight necessary relations (57), (59). We now prove that these are also sufficient. Suppose that any ten functions satisfying these relations are given. Then, on account of (59), the equations (58) will be consistent; hence a function Φ can be determined.* It is then easily verified that, on account of the relations (57), the equations (56) hold.

*The integration constant involved enters additively in Φ. Hence, from (9), the force ϕ, ψ, χ is determined except for a constant factor. This agrees with the result stated in article 3.

In order that a system of curves with properties I, II, *and* III, *that is, a system defined by equations of the form* (54), (54′), (54″), *shall be of the dynamical type, it is necessary and sufficient that the ten functions* (55) *shall satisfy the relations* (57) *and* (59).

27. *Preliminary interpretation.* It remains to interpret the relations (57), (59) geometrically. For any system of curves with properties I, II, III there is associated with each point x, y, z a definite cubic curve Γ, whose equations are given by (53). The association of point and corresponding cubic is fixed by the functions (55). Any relation between the functions is thus equivalent to a restriction on this association. It would in this way be possible to derive a direct geometrical interpretation of the relations (57), (59) which express the character of the association in the dynamical case. However, the result thus obtained is quite complicated and we shall not state it. A simple geometric interpretation will be obtained indirectly in the subsequent discussion of associated S-systems.

28. We derive first a property which is equivalent to two of the eight relations (57). According to article 13 there corresponds to any element (y', z') at a given point a definite straight line. Consider now the particular element (ω_1, ω_2) whose direction is that of the force acting at the point. By taking the axis of abscissas in this direction, we find that the corresponding line is represented by

$$X = 0, \qquad \omega_{1x} Y + \omega_{2x} Z = 3. \tag{60}$$

The perpendicular distance from the given point, here taken as the origin of the X, Y, Z system, to this line is

$$\frac{3}{\sqrt{\omega_{1x}^2 + \omega_{2x}^2}}.$$

On the other hand the radius of curvature of the line of force (10) passing through the given point is, under the assumed conditions,

$$\frac{1}{\sqrt{\omega_{1x}^2 + \omega_{2x}^2}};$$

that is, just one third of the perpendicular distance.

The direction of the straight line (60) is also simply related to the line of force. The osculating plane of the latter is

$$\omega_{2x} Y - \omega_{1x} Z = 0.$$

It is thus perpendicular to the line (60). The results may be stated as follows:

The straight line which corresponds, in accordance with property II, *to a lineal element belonging to a line of force is parallel to the binormal of the*

line of force; and its perpendicular distance from the given point is three times the radius of curvature of the line of force.

This may be stated in purely geometric form by replacing the direction of the force by that of the axis of the pencil considered in property I. It is found that the results hold for all systems in which the functions (55) satisfy the conditions

$$\begin{aligned}\{m_1+m_2\omega_1+m_3\omega_2+\omega_1(l_1\omega_1+l_2\omega_2)\}^2+\{n_1+n_2\omega_1+n_3\omega_2+\omega_2(l_1\omega_1+l_2\omega_2)\}^2\\ =(\omega_{1x}+\omega_1\omega_{1y}+\omega_2\omega_{1z})^2+(\omega_{2x}+\omega_1\omega_{2y}+\omega_2\omega_{2z})^2\\ -\frac{1}{1+\omega_1^2+\omega_2^2}\begin{vmatrix}\omega_1 & \omega_{1x}+\omega_1\omega_{1y}+\omega_2\omega_{1z}\\ \omega_2 & \omega_{2x}+\omega_1\omega_{2y}+\omega_2\omega_{2z}\end{vmatrix},\end{aligned} \tag{61}$$

$$\begin{vmatrix}\omega_{1x}+\omega_1\omega_{1y}+\omega_2\omega_{1z} & m_1+m_2\omega_1+m_3\omega_2+\omega_1(l_1\omega_1+l_2\omega_2)\\ \omega_{1x}+\omega_1\omega_{2y}+\omega_2\omega_{2z} & n_1+n_2\omega_1+n_3\omega_2+\omega_2(l_1\omega_1+l_2\omega_2)\end{vmatrix}=0. \tag{62}$$

These are, in fact, consequences of the relations (57), though of course the converse is not true. The property stated in this article thus holds not only for dynamical but for more general systems. Its addition to I, II, and III will therefore not yield a complete characterization of the dynamical type.

The associated plane systems S.

29. *Definition.* To reach the desired characterization we introduce certain plane systems of curves which may be associated with any quintuply infinite system of space curves.

Consider any plane π. Through each point of π there pass ∞^2 curves of the given system which are tangent to π. Project the differential elements of the third order belonging to these space curves orthogonally upon π, thus obtaining ∞^2 plane differential elements of the third order at the selected point. Applying this process to all points of π, we have a definite system of ∞^4 differential elements of the third order. These elements define a certain differential equation of the third order and thus determine ∞^3 integral curves. These form what we shall term the *associated S-system* in the plane π.

The system S is easily determined analytically. Let the given plane π be taken as the xy plane, and let the equations of the arbitrary quintuply infinite system be

$$\begin{aligned}z''&=g(x,y,z,y',z',y''),\\ y'''&=f(x,y,z,y',z',y'').\end{aligned} \tag{63}$$

Then for the lineal elements contained in π we have

$$z=0, \qquad z'=0.$$

Hence the differential equation of S is

$$y''' = f(x, y, 0, y', 0, y'') = f_0(x, y, y', y''). \tag{64}$$

There will be in all ∞^3 of these triply infinite systems S, namely, one for each plane of space. Any geometric property of these systems is at the same time a property of the original space system; for the former are determined geometrically by the latter.

30. We apply this notion to the dynamical systems and obtain:

PROPERTY IV. *The plane systems S associated with any quintuply infinite system of dynamical trajectories are of the (two-dimensional) dynamical type.*

To prove this it will be sufficient to consider the S-system associated with the xy plane. Applying the method given in the preceding article to the equations (8), or, as is equivalent, to the equations (5) and (6), we find that the system is defined by

$$(\bar{\psi} - y'\bar{\phi})y''' = \begin{vmatrix} 1 & \bar{\phi}_x + y'\bar{\phi}_y \\ y' & \bar{\psi}_x + y'\bar{\psi}_y \end{vmatrix} - 3\bar{\phi}y''^2, \tag{65}$$

where $\bar{\phi}$, $\bar{\psi}$ are functions of x, y obtained from ϕ, ψ by substituting 0 for z This is precisely the equation of the trajectories generated by the plane field of force whose components are $\bar{\phi}$, $\bar{\psi}$.* The force acting in the plane is seen to be derivable by orthogonal projection from the given force (1).

31. The geometrical properties of plane dynamical systems are given in the paper already cited. We shall number the properties as in that paper, but distinguish them from the spatial properties already obtained by attaching a subscript p. As the characteristic set we may take †

$$\mathrm{I}_p,\ \mathrm{II}_p,\ \mathrm{III}_p,\ \mathrm{V}_p,\ \mathrm{VI}_p. \tag{67}$$

Property IV, given in article 30, may thus be restated in purely geometric form as follows:

PROPERTY IV. *The systems S associated with any quintuply infinite system of dynamical trajectories possess the plane properties I_p, II_p, III_p, V_p, VI_p.*

It thus may be broken up into five distinct statements, of which the first, for example, would be: Every S-system is such that the osculating parabolas (constructed at the common point) of the ∞^1 curves passing through a given point in a given direction have their foci located on a circle passing through the given point.

* Cf. *Trajectories*, p. 403, formula (5).

† Loc. cit., p. 417.

32. *Converse of* I_p. The object of the remaining discussion is to show that the addition of property IV to properties I, II, III gives a set which completely characterizes the dynamical type in space. We must therefore examine the effect of adding successively the properties (67) to I, II, III. The first result is this:

If any quintuply infinite system of space curves possesses properties I, II, *then the associated S-systems necessarily possess property* I_p.

This is proved by noting that the condition for I_p is that the differential equation of the plane system shall be of the form *

$$y''' = G(x, y, y')\,y'' + H(x, y, y',)\,y'^2. \tag{68}$$

That the S-system is of this type is seen from the equations (27), which represent the most general system with properties I and II.

It is thus shown that I_p is redundant, in the sense that it is a consequence of preceding spatial properties.

33. *Converse of* II_p, III_p. In order that the system (68) shall have properties II_p, III_p, it is necessary that the G and H shall have the special forms †

$$G = \frac{\lambda y'^2 + \mu y' + \nu}{y' - \omega}, \qquad H = \frac{3}{y' - \omega}, \tag{69}$$

where λ, μ, ν, ω involve x, y only.

We now determine under what conditions a system with properties I, II, and III, that is, a system whose differential equations are of the type defined by (54), (54′), (54″), will have associated S-systems possessing properties II_p, III_p. This is to be the case, of course, for the S-system in every plane.

Consider first the planes parallel to the xy plane. The systems S are then found by substituting 0 for z' and an arbitrary constant for z. By making use of forms (69) in connection with (54″), we find

$$\begin{aligned} m_1 + m_2 y' + l_1 y'^2 &= \lambda y'^2 + \mu y' + \nu, \\ n_1 + n_2 y' &= -\frac{\omega_2(\lambda y'^2 + \mu y' + \nu) + (y' - \omega_1)\omega_2' + \omega_2 \omega_1'}{y' - \omega_1}. \end{aligned} \tag{70}$$

The elimination of λ, μ, ν then gives

$$\begin{vmatrix} y' - \omega_1 & m_1 + \omega_{1x} + (m_2 + \omega_{1y})y' + l_1 y'^2 \\ -\omega_2 & n_1 + \omega_{2x} + (n_2 + \omega_{2y})y' \end{vmatrix} = 0.$$

* *Trajectories*, article 13.

† Loc. cit., articles 14, 15.

This is to be satisfied identically; hence

$$n_2 + \omega_{2y} + l_1\omega_2 = 0,$$

$$\omega_1(n_1 + \omega_{2x}) - \omega_2(m_1 + \omega_{1x}) = 0, \tag{71}$$

$$n_1 + \omega_{2x} - \omega_1(n_2 + \omega_{2y}) + \omega_2(m_2 + \omega_{1y}) = 0.$$

An analogous discussion for the systems S in the planes parallel to the xz plane yields the relations

$$m_3 + \omega_{1z} + l_2\omega_1 = 0,$$

$$\omega_1(n_1 + \omega_{2x}) - \omega_2(m_1 + \omega_{1x}) = 0, \tag{72}$$

$$m_1 + \omega_{1x} + \omega_1(n_3 + \omega_{2z}) - \omega_2(m_3 + \omega_{1z}) = 0.$$

No new relations are found by considering other planes. The relations (71), (72) are seen to reduce to five and to be equivalent to the relations (57).

The only systems of curves with properties I, II, III, I_p, II_p, III_p *are those systems of type* (54), (54′), (54″) *in which the functions* (55) *satisfy the relations* (57).

34. It has been proved incidentally that when II_p, III_p hold for the systems S in two series of parallel planes, say the planes parallel to two of the coördinate planes, they then necessarily hold for all planes of space. Of course it is assumed here, as in the following, that I, II, III hold for the original space system.

35. *Converse of* V_p. It will now be shown that property V_p imposes no additional restrictions. The condition that a plane system defined by the forms (69) shall have this property is *

$$\lambda\omega^2 + \mu\omega + \nu + \omega_x + \omega\omega_y = 0. \tag{73}$$

Applying this to the S-systems in the planes parallel to the xy plane, for which, according to (70),

$$\lambda = l_1, \qquad \mu = m_2, \qquad \nu = m_1, \qquad \omega = \omega_1, \tag{74}$$

we find

$$l_1\omega_1^2 + m_2\omega_1 + m_1 + \omega_{1x} + \omega_1\omega_{1y} = 0. \tag{75}$$

This, however, is seen to be a consequence of the relations (57).

Property V_p *is a consequence of* II_p, III_p *in connection with* I, II, III.

36. *Converse of* VI_p. It will now be shown that property VI_p does impose additional restrictions and that these are precisely equivalent to the relations (59) obtained in article (25).

* *Trajectories*, article 22.

The condition for VI_p is*

$$\lambda_x + \left(\frac{\nu + \omega_x}{\omega}\right)_y = 0. \tag{76}$$

We begin by applying this to the S-systems in the planes parallel to the xy plane. In this case we may use the values given in (74); hence

$$l_{1x} + \left(\frac{m_1 + \omega_{1x}}{\omega_1}\right)_y = 0;$$

which is the third of the equations (59). The corresponding condition for the planes parallel to the xz plane is obtained by replacing y by z, ω_1 by ω_2, l_1 by l_2 and m_1 by n_1. This gives the second of the equations (59).

The discussion for the S-systems in planes parallel to the yz plane is not so simple; for here we can no longer employ x as the independent variable. It is necessary to transform the second of the equations (54) so that say y shall be the independent variable. The result is

$$\frac{d^3z}{dy^3} = \frac{\lambda\left(\frac{dz}{dy}\right)^2 + \mu\frac{dz}{dy} + \nu}{\frac{dz}{dy} - \omega}\,\frac{d^2z}{dy^2} + \frac{3}{\frac{dz}{dy} - \omega}\left(\frac{d^2z}{dy^2}\right)^2, \tag{77}$$

where

$$\lambda = -\frac{m_3}{\omega_1}, \qquad \mu = \frac{n_3 - m_2 - m_1\frac{dx}{dy}}{\omega_1}, \qquad \nu = \frac{n_2 + n_1\frac{dx}{dy}}{\omega_1}, \qquad \omega = \frac{\omega_2}{\omega_1}. \tag{77'}$$

For the S-systems considered, x is an arbitrary constant and $dx/dy = 0$. Thus the condition (76) gives

$$\frac{\partial}{\partial y}\left(-\frac{m_3}{\omega_1}\right) + \frac{\partial}{\partial z}\left(\frac{\frac{n_2}{\omega_1} + \left(\frac{\omega_2}{\omega_1}\right)_\nu}{\frac{\omega_2}{\omega_1}}\right) = 0.$$

By using the relations (57), it may be shown that this takes the simple form

$$l_{1z} - l_{2y} = 0,$$

which is the first of the relations (59).

We may avoid the calculations that would be introduced by the direct consideration of other planes as follows. It has been shown that when the S-systems in planes parallel to the coördinate planes possess property VI_p, then relations (59) are necessarily fulfilled. The relations (57) are already fulfilled on account of properties II_p, III_p. From the theorem of article 25 it then follows that the

* *Trajectories*, p. 415.

original system of space curves is of dynamical type. Hence, by article 30, the systems S for all the planes of space are of dynamical type and hence possess VI_p.

In order that a system with properties I, II, III, II_p, III_p *shall also possess property* VI_p *it is necessary and sufficient that the relations* (59) *shall be fulfilled.*

37. It has been shown incidentally that when VI_p holds for the S-systems constructed in a triply orthogonal series of planes, it then holds for all S-systems.

38. *Restatement of* IV. It will be recalled that the plane systems S were obtained, not by the orthogonal projection of the space curves, but by the orthogonal projection of the differential elements of the third order belonging to these curves. If we consider the ∞^4 space curves of the original quintuply infinite system which touch a given plane π, their orthogonal projections upon that plane would give rise to a system of ∞^4 plane curves. The relation between these curves and those belonging to the system S in the same plane is as follows: the projections of the ∞^2 space curves touching π at a common point O have, at that point, contact of the third order with the ∞^2 S-curves passing through O.

Properties I_p, II_p, III_p have reference to the curves passing through a given point; properties V_p, VI_p relate to certain variations produced by changing that point. Neither set involves differential elements of higher than the third order. Hence it is possible to replace property IV by the following equivalent:

If the ∞^2 *trajectories touching a given plane* π *at a common point* O *are projected orthogonally upon* π, *the plane curves thus obtained possess the properties* I_p, II_p, III_p *and hence also the derived property* IV_p *as stated in the two-dimensional theory.* When the point* O *varies in the plane* π, *the direction associated with it by* II_p *and the conic associated with it by* IV_p *vary in accordance with the restrictions expressed in* V_p *and* VI_p.*

Complete characterization.

39. The main result of our entire discussion may be stated as follows:

In order that a system of ∞^5 *space curves, of which* ∞^1 *pass through each point in each direction, shall be identifiable with the system of trajectories due to a (positional) field of force*

$$\frac{d^2x}{dt^2} = \phi(x, y, z), \qquad \frac{d^2y}{dt^2} = \psi(x, y, z), \qquad \frac{d^2z}{dt^2} = \chi(x, y, z),$$

it is necessary and sufficient that it shall have the following purely geometric properties:

I. *The osculating planes of the* ∞^2 *curves passing through a given point form a pencil; that is, all the planes pass through a fixed direction.*

* See *Trajectories.*

II. *The osculating spheres of the* ∞^1 *curves passing through a given point in a given direction form a pencil; their centers thus lie on a straight line.*

III. *The straight lines which correspond, in accordance with* II, *to all the directions at a given point, form a congruence, of order one and class three, consisting of the secants of a twisted cubic curve; which cubic passes through the given point in the direction fixed by property* I.

IV. *The associated systems* S, *constructed as in article* 29, *have the characteristic properties of plane dynamical systems.*

These four properties are independent in the sense that no one can be derived from those which precede it.* The last one, IV, may be replaced by weaker requirements: of the component parts it is sufficient to retain II_p, III_p, VI_p, and of all S-systems it is sufficient to consider those constructed in a triply orthogonal series of planes.

40. From article 22 it is seen that a complete geometric characterization of the trajectories produced by a *conservative force* is obtained by adding to III in the preceding set the requirement that, for every point of space, the corresponding cubic curve shall be of the rectangular species.

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* The question of absolute independence is left open. It may be that the other properties are consequences of IV.

A CLASS OF PERIODIC ORBITS OF AN INFINITESIMAL BODY SUBJECT TO THE ATTRACTION OF n FINITE BODIES*

BY

WILLIAM RAYMOND LONGLEY

Introduction.

The question of periodic solutions of a set of differential equations with application to the problem of three bodies has been treated by POINCARÉ,† and the idea of analytic continuation which he developed will be employed in this paper to prove the existence of the solutions under consideration.

Suppose that n finite bodies move in one plane and that their coördinates with reference to one of the bodies, M, as origin are known functions of the time. At any epoch t_0 the bodies form a certain geometrical figure and have certain relative components of velocity. Suppose that at the epoch $t_0 + T_1$ the bodies form again the same geometrical figure and have the same relative components of velocity; it is said that the motion is periodic with the period T_1.

A particle P is supposed to move in the plane of the motion of the finite bodies, subject to their newtonian attraction. In the differential equations of motion of the particle it is convenient to divide the terms into two classes: (1) those which are due to the attraction of M alone, and, (2) those which are due to the attraction of the other bodies. A parameter μ is introduced as a factor of all the terms of the second class, so that when $\mu = 0$ the differential equations represent the motion of P when subject to the attraction of only one body, M. In this case the solution is known and the conic in which the particle moves is called the *undisturbed orbit.*

If there exists a periodic solution of the differential equations when $\mu \neq 0$,‡ the period T of the solution must be a multiple of T_1, $T = qT_1$. Suppose the period is assigned in advance. When $\mu = 0$ a periodic solution is known. The initial conditions may be determined so that the period T_2 of this solution is a submultiple of the assigned period, that is, $pT_2 = qT_1 = T$, where p and q are

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† Bulletin Astronomique, vol. 1 (1884), p. 65; Acta Mathematica, vol. 13 (1890), p. 5; *Les méthodes nouvelles de la mécanique céleste,* vol. 1, chap. 3 (1892).

‡ The analysis employed in this paper is valid for sufficiently small values of the parameters involved. All statements concerning the properties of the solutions for $\mu \neq 0$ refer only to values of μ so small that the solutions are known to be valid.

integers. There exists, therefore, a solution of the differential equations for $\mu = 0$, which has the assigned period. Does this solution persist, in the sense of analytic continuation, for values of μ different from zero but sufficiently small? This question is here investigated by the method of POINCARÉ. Three cases present themselves for consideration:

I. When the orbit of the particle is symmetrical with respect to a line.

II. When there exists a uniform integral of the differential equations of motion.

III. When no uniform integral exists.

I. The first case may occur as a particular sub-case of either the second or the third; but, if the differential equations admit a symmetrical solution, the proof of the existence of a symmetrical periodic orbit does not depend upon the consideration of the integral, and may be most conveniently treated as a distinct case.

If a particle is subject to the attraction of two finite bodies, there are always initial conditions under which the orbit is symmetrical with respect to the line joining the finite bodies, and with respect to the time of crossing this line. If the body M_1 revolves about M in a circle, the condition for a symmetrical orbit of the particle P is that P shall cross the radius vector of M_1 orthogonally at any time. The orbits of HILL,* DARWIN,† and MOULTON‡ are of this type. If the orbit of M_1 is an ellipse, the condition is that P shall cross the radius vector orthogonally when M_1 is at an apse.§ The property of the symmetry of the orbit of P with respect to a line rotating with an angular velocity which is a function of t^2, holds for some cases in which the particle is subject to the attraction of more than two finite bodies. When the motion of n finite bodies is such that they form a geometrical figure having always the same shape, in which equal masses are symmetrically situated with respect to a line of the figure, the particle may be started so that its orbit will be symmetrical.‖ An example of another type of the motion of the finite bodies, which is such that the orbit of the particle may be symmetrical, is given in the last section of this paper in the third numerical problem.

II. The case where a uniform integral exists occurs when the finite bodies move in circles about M with the same angular velocity N. The bodies form a fixed geometrical configuration in a plane which rotates about the origin with

* American Journal of Mathematics, vol. 1 (1878), p. 245.

† Acta Mathematica, vol. 21 (1897), p. 99.

‡ Transactions of the American Mathematical Society, vol. 7 (1906), p. 537.

§ This case has been treated in an unpublished paper by W. D. MACMILLAN presented to the Society, April 14, 1906.

‖ Examples are furnished by the equilateral triangular solution of Lagrange, when two of the bodies are equal, and by many particular solutions of the problem of n bodies. See W. R. LONGLEY, *Some particular solutions in the problem of n bodies*, Bulletin of the American Mathematical Society, vol. 13 (1907), p. 324.

the angular velocity N. When the coördinates of the particle are referred to axes in the rotating plane, the differential equations of motion do not contain the time explicitly, and the period of the solution may be chosen arbitrarily except for limitations which are necessary to assure the convergence of the series employed.

III. The third case occurs when the finite bodies do not move in circular orbits about M. The parameter μ is introduced into the differential equations in such a way that the proof of the existence of periodic orbits of the particle is made to depend upon the terms which are due to the attraction of only one of the perturbing bodies, M_1. The motion of M_1 is subject to mild restrictions, namely, that the radius vector shall be given by an expression involving only cosines of multiples of the time, and that the expressions for the longitude shall involve only sines of multiples of the time. The only restriction upon the motion of the other bodies is that it shall be periodic. The treatment is sufficiently general to permit applications in the problems presented by the motions of the solar system. For example, suppose that P is a satellite of one of the planets M, and that M_1 is the sun. The conditions upon the motion of M_1 are fulfilled if we neglect the perturbations of the other planets upon M. If we neglect the inclinations of the orbits of the other planets, and suppose that their motion is periodic, it is possible by the methods given below to treat the periodic motion of the satellite in the plane of the planetary orbit when subject to the attraction of the sun and all the planets.

It is shown that when the period of the motion of the particle is assigned, there exist two and only two real orbits of this analytic type having the prescribed period. In one of the orbits the motion is direct, and in the other it is retrograde. If symmetrical orbits exist, there are no unsymmetrical periodic orbits. It is shown that, in the analysis employed, the eccentricity of the undisturbed orbit must be zero. A method is given for constructing the periodic solutions to any desired degree of accuracy by processes which involve only algebraic computation.

For convenience and clearness the details of the analysis are carried out for a special example of the motion of the finite bodies, namely, for three finite bodies which move according to the equilateral triangular solution of Lagrange. This example illustrates the three cases which may occur, and only slight changes in the details are necessary in order to apply the treatment to the more general examples previously mentioned.

Some numerical examples are given at the end of this paper with drawings of the orbits which have been computed.

Existence of Periodic Orbits.

The differential equations of motion. Let M, M_1, and M_2 denote the masses of three finite bodies moving according to the lagrangian equilateral triangular solution. With reference to M as origin and an axis having a fixed direction in space, let the coördinates of M_1, M_2, and the particle P be respectively (R_1, V_1), $(R_2\ V_2)$, and (r, v). The coördinates of the bodies are expressed in terms of the time as follows:

$$(1)\qquad \begin{aligned} R_1 = R_2 = AR &= A\left\{1 - \epsilon\cos Nt - \frac{\epsilon^2}{2}(\cos 2Nt - 1) + \cdots\right\},\\ V_1 - \varpi_1 = V_2 - \varpi_2 = V &= Nt + 2\epsilon\sin Nt + \tfrac{5}{4}\epsilon^2\sin 2Nt + \cdots, \end{aligned}$$

where

$$N^2A^3 = k^2(M + M_1 + M_2).$$

The finite bodies are at apses of their respective orbits at $t = 0$; N is the mean angular motion, A the major semi-axis, ϵ the eccentricity, k is a constant depending upon the units employed; and ϖ_1, ϖ_2 are constants such that $\varpi_2 - \varpi_1 = \pi/3$.

The differential equations of motion of P are

$$(2)\qquad \begin{aligned} \frac{d^2r}{dt^2} - r\left(\frac{dv}{dt}\right)^2 + \frac{k^2M}{r^2} &= \frac{\partial\Omega}{\partial r},\\ r\frac{d^2v}{dt^2} + 2\frac{dr}{dt}\frac{dv}{dt} &= \frac{1}{r}\frac{\partial\Omega}{\partial v}, \end{aligned}$$

where

$$\Omega = k^2\left[\frac{M_1}{r_1} + \frac{M_2}{r_2} - \frac{M_1}{R_1^2}r\cos(v - V_1) - \frac{M_2}{R_2^2}r\cos(v - V_2)\right],$$

$$(3)\qquad \begin{aligned} r_1 &= \sqrt{r^2 + R_1^2 - 2rR_1\cos(v - V_1)},\\ r_2 &= \sqrt{r^2 + R_2^2 - 2rR_2\cos(v - V_2)}. \end{aligned}$$

Let us define m and a by the relations

$$(4)\qquad m\nu = N, \qquad \nu^2a^3 = k^2M,$$

where ν is the mean angular velocity of P in the undisturbed orbit.

The period of the motion of the particle in the undisturbed orbit is $2\pi/\nu$. The period of the motion of the finite bodies is $2\pi/N$. In order that the periods shall be commensurable, ν is determined by the relation $2p\pi/\nu = 2q\pi/N$, where p and q are integers. This restriction is not necessary, however, when $\epsilon = 0$.

The next step in obtaining the final form of the differential equations is to refer the motion to a plane rotating with the angular velocity of the finite bodies. If $\epsilon = 0$, the coördinates of M_1 and M_2 become constants, and ν may be

chosen arbitrarily. The motion is referred to an axis rotating with the angular velocity of the finite bodies by the substitution $v = w + V$. It is convenient to eliminate the factors depending upon the units employed by, the substitution

$$r = a\rho, \qquad \nu t = \tau.$$

Hence Nt becomes $m\tau$. We may expand Ω as a power series in $a\rho/AR$ as follows:

$$\Omega = \frac{k^2 M_1}{AR}\left[1 + \tfrac{1}{4}\left(\frac{a\rho}{AR}\right)^2\left\{1 + 3\cos 2(w - \varpi_1)\right\}\right.$$
$$\left. + \tfrac{1}{8}\left(\frac{a\rho}{AR}\right)^3\left\{3\cos(w - \varpi_1) + 5\cos 3(w - \varpi_1)\right\} + \cdots\right]$$
$$+ \frac{k^2 M_2}{AR}\left[1 + \tfrac{1}{4}\left(\frac{a\rho}{AR}\right)^2\left\{1 + 3\cos 2(w - \varpi_2)\right\}\right.$$
$$\left. + \tfrac{1}{8}\left(\frac{a\rho}{AR}\right)^3\left\{3\cos(w - \varpi_2) + 5\cos 3(w - \varpi_2)\right\} + \cdots\right].$$

Let λ_1 and λ_2 be defined by the relations

$$M_1 = \lambda_1(M_1 + M_2), \qquad M_2 = \lambda_2(M_1 + M_2). \tag{5}$$

From equations (1) we have

$$\frac{k^2(M_1 + M_2)}{A^3} = \frac{N^2}{1 + \dfrac{M}{M_1 + M_2}}.$$

On setting

$$\frac{1}{1 + \dfrac{M}{M_1 + M_2}} = K,$$

it follows that

$$\frac{1}{a}\frac{\partial\Omega}{\partial(a\rho)} = \frac{Km^2\rho}{R^3}\left[\frac{\lambda_1}{2}\left\{1 + 3\cos 2(w - \varpi_1)\right\}\right.$$
$$+ \frac{3\lambda_1}{8}\left(\frac{a\rho}{AR}\right)\left\{3\cos(w - \varpi_1) + 5\cos 3(w - \varpi_1)\right\} + \cdots$$
$$+ \frac{\lambda_2}{2}\left\{1 + 3\cos 2(w - \varpi_2)\right\} \tag{6}$$
$$\left. + \frac{3\lambda_2}{8}\left(\frac{a\rho}{AR}\right)\left\{3\cos(w - \varpi_2) + 5\cos 3(w - \varpi_2)\right\} + \cdots\right],$$

$$
\begin{aligned}
\frac{1}{a^2\rho}\frac{\partial\Omega}{\partial w} = -\frac{Km^2\rho}{R^3}\Big[& \frac{3\lambda_1}{2}\sin 2(w-\varpi_1) \\
& + \frac{3\lambda_1}{8}\left(\frac{a\rho}{AR}\right)\left\{\sin(w-\varpi_1)+5\sin 3(w-\varpi_1)\right\}+\cdots \\
& + \frac{3\lambda_2}{2}\sin 2(w-\varpi_2) \\
& + \frac{3\lambda_2}{8}\left(\frac{a\rho}{AR}\right)\left\{\sin(w-\varpi_2)+5\sin 3(w-\varpi_2)\right\}+\cdots\Big].
\end{aligned} \tag{6}
$$

The differential equations of motion become

$$
\begin{aligned}
& \frac{d^2\rho}{d\tau^2}-\rho\left(\frac{dw}{d\tau}+\frac{dV}{d\tau}\right)^2+\frac{1}{\rho^2}=\frac{1}{a}\frac{\partial\Omega}{\partial(a\rho)}, \\
& \rho\left(\frac{d^2w}{d\tau^2}+\frac{d^2V}{d\tau^2}\right)+2\frac{d\rho}{d\tau}\left(\frac{dw}{d\tau}+\frac{dV}{d\tau}\right)=\frac{1}{a^2\rho}\frac{\partial\Omega}{\partial w}.
\end{aligned} \tag{7}
$$

Case I. Symmetrical solutions. Suppose the masses M_1 and M_2 are equal and that the w-axis passes through the center of gravity of the system. These conditions are expressed by the relations,

$$
\lambda_1=\lambda_2, \qquad \varpi_1=-\frac{\pi}{6}, \qquad \varpi_2=\frac{\pi}{6}. \tag{8}
$$

It follows that the first of equations (7) involves only *cosines* of multiples of w, and the second involves only *sines* of multiples of w. Since $dV/d\tau$ and R are even functions of τ, it follows that the equations are unchanged if ρ is left unchanged while w and τ are changed into $-w$ and $-\tau$ respectively.* When subject to the restrictions (8), let the differential equations be denoted by (7_I).†

Suppose that

$$
\rho=\psi_1(\tau), \qquad w=\psi_2(\tau), \tag{9}
$$

is a solution of (7_I) such that

$$
\left.\frac{d\rho}{d\tau}\right|_{\tau=p\pi}=0, \qquad w(p\pi)-p\pi=0. \tag{10}
$$

The differential equations are periodic in w and τ, and M_1 and M_2 are at apses at $\tau=p\pi$. It follows from the form of the differential equations that ψ_1 is an even function and ψ_2 is an odd function of τ. Hence if the particle crosses the w-axis when the finite bodies are at apses of their respective orbits, the orbit of P is symmetrical with respect to the axis and the time of crossing. If the orbit is symmetrical with respect to two epochs, for example, $\tau_0=0$ and $\tau_1=p\pi$, it is periodic with the period $2p\pi$. If the solution (9) is symmetrical with

* In the more general cases in which symmetrical solutions occur it may happen that the first equation contains also sines of multiples of w multiplied by odd functions of τ, and that the second equation contains also cosines of multiples of w multiplied by odd functions of τ.

† Throughout this paper the subscripts I, II, III refer to the case I, II, III, respectively.

respect to the epoch $\tau = 0$, the conditions (10) are necessary and sufficient for it to be periodic with the period $2p\pi$.

We will now investigate the existence of symmetrical periodic orbits which have the period $2p\pi$. The terms of equations (7_I) are periodic in τ and analytic in m; but, when expanded in this parameter, the coefficients of the various powers of m contain non-periodic terms. It is convenient in this discussion of periodic solutions to avoid such terms in the differential equations, and this is accomplished by a generalization of the parameter m.* Consider the set of differential equations,

$$\begin{aligned} &\frac{d^2\rho}{d\tau^2} - \rho\left(\frac{dw}{d\tau} + \mu\frac{dU}{d\tau}\right)^2 + \frac{1}{\rho^2} = \mu^2 f_1, \\ &\rho\left(\frac{d^2w}{d\tau^2} + \mu\frac{d^2U}{d\tau^2}\right) + 2\frac{d\rho}{d\tau}\left(\frac{dw}{d\tau} + \mu\frac{dU}{d\tau}\right) = \mu^2 g_1, \end{aligned} \tag{11}$$

where

$$U = \frac{\mu}{m}\ V = \frac{\mu}{m}\left\{m\tau + 2\epsilon\sin m\tau + \tfrac{5}{4}\epsilon^2\sin 2m\tau + \cdots\right\}$$

and $\mu^2 f_1$ and $\mu^2 g_1$ are obtained from $1/a\cdot\partial\Omega/\partial(a\rho)$ and $1/a^2\rho\cdot\partial\Omega/\partial w$ respectively by writing μ^2 for the factor m^2, and by making the substitution $a/A = \eta\mu$, where η is a numerical constant.

Equations (11) do not represent any physical problem except for a special value of μ, namely, $\mu = m$, and are to be studied from a mathematical point of view. They may be solved for $\rho - 1$ and w as power series in μ. For a preassigned arbitrary interval for τ the series are convergent for all values of μ numerically less than $\bar{\mu}$,† and satisfy the differential equations identically in μ. Therefore, if in the solution μ is given a value numerically equal to $m (m < \mu)$, the solution satisfies the differential equations (7_I) and represents the physical problem under consideration. It follows from the form in which m enters the differential equations that $\bar{\mu}$ is independent of m.

When $\mu = 0$ the equations are of the form occurring in the problem of two bodies, and a symmetrical solution having the required period is known, namely,

$$\begin{aligned} &\rho = 1 - \bar{e}\cos E, \\ &w = \arccos\left(\frac{\cos E - \bar{e}}{1 - \bar{e}\cos E}\right) = \arcsin\left(\frac{\sqrt{1-\bar{e}^2}\sin E}{1 - \bar{e}\cos E}\right), \text{‡} \end{aligned} \tag{12}$$

where E is defined by the relation

$$\tau = E - \bar{e}\sin E,$$

* F. R. MOULTON, *On certain rigorous methods of treating problems in celestial mechanics*. The Decennial Publications of The University of Chicago, vol. 3 (1902), p. 126.

† POINCARÉ, *Les méthodes nouvelles de la mécanique céleste*, vol. 1 (1892), p. 58.

‡ The positive sign is to be taken with the radical.

and $\bar{e}$ is an arbitrary constant less than unity. The initial conditions for $\tau = 0$ are

$$(13) \qquad \rho = 1 - \bar{e}, \qquad \frac{d\rho}{d\tau} = 0, \qquad w = 0, \qquad \frac{dw}{d\tau} = \frac{\sqrt{1-\bar{e}^2}}{(1-\bar{e})^2}.$$

Consider the solution for values of μ different from zero but sufficiently small, and let the initial conditions for $\tau = 0$ be

$$(14) \qquad \rho = 1 - \bar{e} + \beta = (1+\alpha)[1-(\bar{e}+e)],^* \qquad \frac{d\rho}{d\tau} = 0, \qquad w = 0,$$
$$\frac{dw}{d\tau} = \frac{\sqrt{1-\bar{e}^2}}{(1-\bar{e})^2} + \gamma = \frac{\sqrt{1-(\bar{e}+e)^2}}{[1-(\bar{e}+e)]^2} - \mu \left.\frac{dU}{d\tau}\right|_{\tau=0}.$$

The solution is symmetrical with respect to the epoch $\tau = 0$, and may be expressed as power series in α, e, and μ, which are convergent for an interval in τ including the interval 0 to $p\pi$, if the parameters are sufficiently small. If α and e can be determined in terms of μ, vanishing with μ, so that the conditions (10) are satisfied, then the solution will be periodic with the period $2p\pi$. All terms of the solution which are independent of μ^2 may be obtained from the two body problem by making the substitution $w = u - \mu U$.

These terms are given in finite form by the expressions

$$\rho = (1+\alpha)[1-(\bar{e}+e)\cos E],$$
$$u = \arccos\left(\frac{\cos E - (\bar{e}+e)}{1-(\bar{e}+e)\cos E}\right) = \arcsin\left(\frac{\sqrt{1-(\bar{e}+e)^2}\sin E}{1-(\bar{e}+e)\cos E}\right),$$

where E is defined by the relation

$$\frac{\tau}{(1+\alpha)^{\frac{3}{2}}} = E - (\bar{e}+e)\sin E.$$

Returning to the variable w, writing the terms in α and e as power series by Taylor's expansion, and applying the conditions (10), we obtain the equations

$$(15) \qquad 0 = -\tfrac{3}{2}p\pi\frac{\bar{e}}{(1-\bar{e})^2}\alpha - \tfrac{3}{4}p\pi\frac{1+\bar{e}}{(1-\bar{e})^3}\alpha e + \cdots,$$
$$0 = -\tfrac{3}{2}p\pi\frac{\sqrt{1-\bar{e}^2}}{(1-\bar{e})^2}\alpha - p\pi\mu + \cdots.$$

It follows from the known properties of the series that there are no terms in e alone, and there are no terms involving μ to the first degree except the term

* The introduction of the very convenient parameters α and e, instead of the additive increments β and γ, is due to Professor F. R. MOULTON, these Transactions, loc. cit., from whom I have received many suggestions and much valuable criticism.

$-p\pi\mu$. Hence the second of equations (15) may be solved for α as a power series in e and μ in which μ is contained as a factor:

$$\alpha = \mu\left\{-\tfrac{3}{8}\frac{(1-\bar{e})^2}{\sqrt{1-\bar{e}^2}}+\cdots\right\}. \tag{16}$$

When this value of α is substituted in the first equation, a factor μ may be divided out, leaving

$$0 = p\pi\frac{\bar{e}}{\sqrt{1-\bar{e}^2}}+\tfrac{1}{2}p\pi\frac{1+\bar{e}}{(1-\bar{e})\sqrt{1-\bar{e}^2}}e+\cdots.$$

This equation may be solved for e as a power series in μ, which vanishes with μ, if and only if $\bar{e}=0$, and the result may then be used to eliminate e from equation (16). Since only those solutions are under consideration which are the analytic continuations with respect to μ of those for $\mu = 0$, the condition $\bar{e} = 0$ must be imposed. *The condition $\bar{e} = 0$ means that the undisturbed orbit must be circular.*

The initial conditions (14) have been determined uniquely as power series in μ, convergent for sufficiently small values of μ, so that the conditions of periodicity are satisfied. When the values of α and e thus found are substituted in the solution, we have $\rho - 1$ and w represented as power series in μ alone, and the solution is periodic with the period $2p\pi$.

Suppose $\mu = m$ and consider the physical interpretation of the solution. The period of the motion of the finite bodies is $2\pi/m = 2p\pi/q$, where p and q are integers which may be chosen arbitrarily except for the condition that q/p must be sufficiently small. The period of the solution is $2p\pi$. Hence during this period the finite bodies make q revolutions, the angle w is increased by $2p\pi$, and therefore the particle makes $q + p$ revolutions with reference to fixed axes. If p and q have the same signs, that is, if m is positive, the motion of the particle is direct. If m is negative, the motion is retrograde. For a given value of m there exists one and only one real symmetrical periodic solution of the differential equations. Hence for a given period there exist two and only two real symmetrical orbits of the particle with the required period. In one the motion is direct, and in the other it is retrograde. All distinct orbits are obtained when p and q are prime to each other.

The initial conditions were limited so that the solutions should be symmetrical. When this condition is not imposed, and whether or not the differential equations admit symmetrical solutions, it will be shown in the next two sections that, for a preassigned period, there exist two and only two real periodic orbits, in one of which the motion is direct and in the other retrograde. Hence, if symmetrical periodic orbits exist, there are no unsymmetrical periodic orbits of this type.

The existence proof was made for a particular example, but did not depend upon the right hand members of the differential equations and consequently holds for any case in which the orbits of the perturbing bodies are such that the differential equations admit symmetrical solutions. The only variations which it is necessary to make in the details for the other cases, are in the particular manner in which the parameter μ is introduced, and in the choice of the rotating axis. It may happen in some cases that there exist several possible axes of symmetry.

Case II. *When a uniform integral exists.* Suppose that M_1 and M_2 are not in general equal and that they move in circles about M with the w-axis passing through M_1. These conditions are expressed by the relations

$$\epsilon = 0, \qquad \varpi_1 = 0, \qquad \varpi_2 = \frac{\pi}{3}. \tag{17}$$

When subject to the conditions (17) let the differential equations of motion be denoted by (7_{II}).

In this case it is convenient to introduce the parameter μ by the relations

$$m = \mu, \qquad \lambda_2 = \lambda\mu, \qquad \frac{a}{A} = \eta\mu, \tag{18}$$

for all powers of a/A higher than the *first*. Here it is not necessary to generalize the parameter m since R and $dV/d\tau$ are constants. By relating λ_2 and μ, the existence proof is made to depend only upon those terms of the differential equations which involve λ_1. The generalization of the parameter a/A is merely for convenience in having finite expressions in the equations which determine the coefficients at the various steps in the solution.

The differential equations become

$$\begin{aligned} &\frac{d^2\rho}{d\tau^2} - \rho\left(\frac{dw}{d\tau} + \mu\right)^2 + \frac{1}{\rho^2} = \mu^2 f_{\text{II}}, \\ &\rho\frac{d^2w}{d\tau^2} + 2\frac{d\rho}{d\tau}\left(\frac{dw}{d\tau} + \mu\right) = \mu^2 g_{\text{II}}, \end{aligned} \tag{19}$$

where

$$f_{\text{II}} = K\rho\left[\lambda_1\left\{\tfrac{1}{2}(1 + 3\cos 2w) + \tfrac{3}{8}\frac{a}{A}\rho(3\cos w + 5\cos 3w)\right\} + \cdots \mu\,\text{func}\,(\rho, \mu, \cos jw, \sin jw)\right],$$

$$g_{\text{II}} = -K\rho\left[\lambda_1\left\{\tfrac{3}{2}\sin 2w + \tfrac{3}{8}\frac{a}{A}\rho(\sin w + 5\sin 3w)\right\} + \cdots \mu\,\text{func}\,(\rho, \mu, \sin jw, \cos jw)\right].$$

Equations (19) are periodic in w with period 2π and do not involve τ explicitly. Suppose that

$$\rho = \psi_1(\tau), \qquad w = \psi_2(\tau),$$

is a solution. The necessary and sufficient conditions that the solution shall be periodic with the period $2p\pi$ are

$$\begin{aligned} &\psi_1(2p\pi) = \psi_1(0), \qquad \left.\frac{d\psi_1}{d\tau}\right|_{2p\pi} = \left.\frac{d\psi_1}{d\tau}\right|_0, \\ &\psi_2(2p\pi) - 2p\pi = \psi_2(0), \qquad \left.\frac{d\psi_2}{d\tau}\right|_{2p\pi} = \left.\frac{d\psi_2}{d\tau}\right|_0. \end{aligned} \tag{20}$$

When $\mu = 0$ a periodic solution is known,* namely,

$$\rho = 1, \qquad w = \tau.$$

The initial conditions for $\tau = 0$ are

$$\rho = 1, \qquad \frac{d\rho}{d\tau} = 0, \qquad w = 0, \qquad \frac{dw}{d\tau} = 1.$$

For μ different from zero, but sufficiently small, consider the solution of equations (19) subject to the initial conditions

$$\begin{aligned} &\rho = 1 + \beta_1 = (1+\alpha)(1 - e\cos\theta), \\ &\frac{d\rho}{d\tau} = \beta_2 = \frac{e\sin\theta}{\sqrt{1+\alpha}(1 - e\cos\theta)}, \\ &w = \beta_3 = \arccos\left[\frac{\cos\theta - e}{1 - e\cos\theta}\right] - \arccos\left[\frac{\cos(\theta-\phi) - e}{1 - e\cos(\theta-\phi)}\right], \\ &\frac{dw}{d\tau} = 1 + \beta_4 = \frac{\sqrt{1-e^2}}{(1+\alpha)^{\frac{3}{2}}(1 - e\cos\theta)^2} - \mu. \end{aligned} \tag{21}$$

The parameters, β_1, β_2, β_3, β_4, which are additive increments to the initial conditions of undisturbed motion, are inconvenient in the discussion which follows. In terms of the parameters α, e, θ, ϕ, the properties of the solution are well known, and the conditions of periodicity can be easily discussed. The geometrical meaning of the angles θ and ϕ is shown in Fig. 1.

We may now write, in finite form, those terms of the solution which do not contain μ, namely,

$$\begin{aligned} &r = (1+\alpha)(1 - e\cos E), \\ &\cos(w + W_1 - W_0) = \frac{\cos E - e}{1 - e\cos E}, \\ &\sin(w + W_1 - W_0) = \frac{\sqrt{1-e^2}\sin E}{1 - e\cos E}, \end{aligned} \tag{22}$$

* As in the case of symmetrical orbits, no greater generality is obtained by assuming that the undisturbed orbit has an eccentricity different from zero. Therefore we will assume here that $\bar{e} = 0$.

where E is defined by the relation

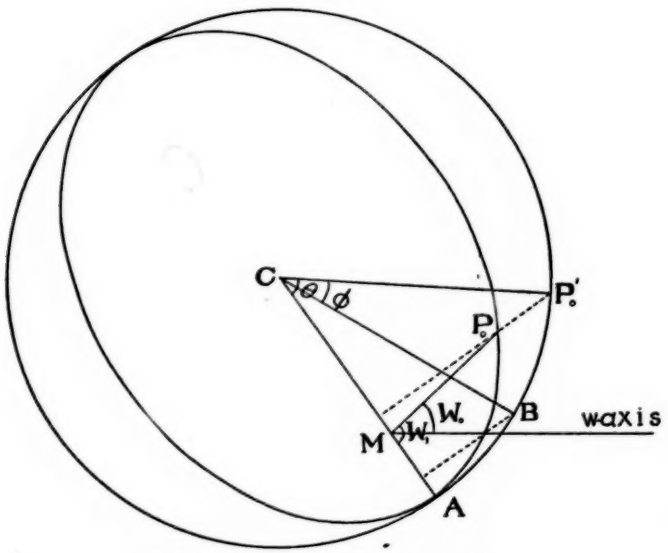
$$\frac{\tau}{(1+\alpha)^{\frac{3}{2}}} + \theta - e\sin\theta = E - e\sin E.$$

The only term of the solution which involves μ to the first degree is $-\tau\mu$, which occurs in the expression for w. To find the terms in μ^2 and $\mu^2\phi$ we may write

$$\rho = 1 + \rho_2\mu^2 + \bar{\rho}\mu^2\phi + \cdots,$$

$$w = \tau - \tau\mu + \phi + w_2\mu^2 + \bar{w}\mu^2\phi + \cdots.$$

FIG. 1.



P_0 is the position of the particle at $\tau = 0$.
W_0 is the longitude of the particle at $\tau = 0$.
$W_1 - W_0$ is the longitude of perihelion.

On substituting these expressions in the differential equations, there results for the determination of ρ_2 and w_2 the following set of equations,

$$\frac{d^2\rho_2}{d\tau^2} - 2\frac{dw_2}{d\tau} - 3\rho_2 = \frac{K\lambda_1}{2}(1 + 3\cos 2\tau) + \frac{3K\lambda_1}{8}\frac{a}{A}(3\cos\tau + 5\cos 3\tau),$$

$$\frac{d^2w_2}{d\tau^2} + 2\frac{d\rho_2}{d\tau} = -\frac{3K\lambda_1}{2}\sin 2\tau - \frac{3K\lambda_1}{8}\frac{a}{A}(\sin\tau + 5\sin 3\tau).$$

It follows that

$$\rho_2 = K\lambda_1\left[-1-2\frac{a}{A}+\left(2+\tfrac{153}{64}\frac{a}{A}\right)\cos\tau + \tfrac{15}{16}\frac{a}{A}\tau\sin\tau-\cos 2\tau-\tfrac{25}{64}\frac{a}{A}\cos 3\tau\right],$$

$$w_2 = K\lambda_1\left[\left(\tfrac{5}{4}+3\frac{a}{A}\right)\tau-\left(4+\tfrac{201}{32}\frac{a}{A}\right)\sin\tau + \tfrac{15}{8}\frac{a}{A}\tau\cos\tau+\tfrac{11}{8}\sin 2\tau+\tfrac{1385}{32}\frac{a}{A}\sin 3\tau\right].$$

By a similar computation it is shown that

$$\bar\rho = K\lambda_1\left[-\left(4+\tfrac{2885}{64}\frac{a}{A}\right)\sin\tau+\tfrac{15}{16}\frac{a}{A}\tau\cos\tau+2\sin 2\tau+\tfrac{75}{64}\frac{a}{A}\sin 3\tau\right],$$

$$\bar w = K\lambda_1\left[\tfrac{21}{4}+\tfrac{2885}{32}\frac{a}{A}-\left(8+\tfrac{3383}{32}\frac{a}{A}\right)\cos\tau - \tfrac{15}{8}\frac{a}{A}\tau\sin\tau+\tfrac{11}{4}\cos 2\tau+\tfrac{455}{32}\frac{a}{A}\cos 3\tau\right].$$

The terms independent of μ are obtained from equations (22) by Taylor's expansion, and the solution becomes

$$\begin{aligned}\rho &= 1+\alpha-e\cos\tau-\alpha e(\cos\tau-\tfrac{3}{2}\tau\sin\tau)+e\theta\sin\tau \\ &\quad+\alpha e\theta(\sin\tau-\tfrac{3}{2}\tau\cos\tau)+\rho_2\mu^2+\bar\rho\mu^2\phi+\cdots, \\ w &= \tau-\tfrac{3}{2}\tau\alpha+2e\sin\tau+\phi-\tau\mu-3\alpha e\tau\cos\tau-e\theta(1-2\cos\tau) \\ &\quad+e\phi+\alpha e\theta(5\cos\tau+3\tau\sin\tau)+w_2\mu^2+\bar w\mu^2\phi+\cdots.\end{aligned} \tag{23}$$

Applying the conditions (20) that the solution shall be periodic, we have

$$\begin{aligned}0 &= -3p\pi\alpha e\theta+\tfrac{15}{8}\frac{a}{A}p\pi\mu^2\phi+\cdots, && (a)\\ 0 &= 3p\pi\alpha e+\tfrac{15}{8}\frac{a}{A}p\pi\mu^2+\cdots, && (b)\\ 0 &= -3p\pi\alpha-2p\pi\mu-6p\pi\alpha e+\cdots, && (c)\\ 0 &= 6p\pi\alpha e\theta-\tfrac{15}{4}\frac{a}{A}p\pi\mu^2\phi+\cdots. && (d)\end{aligned} \tag{24}$$

The conditions (24) involve the four quantities α, e, θ, ϕ, and, if independent, would determine them in terms of μ. But the differential equations (19) do not involve τ explicitly and hence admit the integral of JACOBI. This fur-

nishes a relation of the type

$$F(\alpha, e, \theta, \phi, \mu) = \text{constant},$$

and equations (24) are not independent.* It follows that if (a), (b), and (c) are solved for the three quantities α, e and θ in terms of μ and ϕ, and the results substituted in (d), the equation is satisfied identically in ϕ. In this problem the dynamical interpretation is simple. Since the finite bodies move in circles the origin of time is arbitrary. The most convenient choice is $\tau = 0$ when $w = 0$, which is equivalent to choosing $\phi = 0$.

Consider the solution of equations (a), (b), and (c) for α, e, and θ. The equations have the following properties:

i. There are no terms independent of α and μ. This follows from the fact that, in the two-body problem, the period does not depend upon e and θ.

ii. There are no terms involving μ to the first degree except the one term $-2p\pi\mu$ which occurs in (c).

iii. There are no terms in θ independent of e, since θ does not enter the initial conditions independently of e.

It follows from these properties and the particular form of the first terms of the equations that α, e, and θ may be determined uniquely as power series in μ by the following steps:

(1) From (c) we obtain

$$\alpha = \mu\left[-\tfrac{2}{3} + \cdots + \text{func}\,(\mu, e, \theta)\right].$$

(2) This value of α when substituted in (b) permits a factor μ to be divided out. Then we may solve the result for e as a power series in μ and θ, in which μ is contained as a factor,

$$e = \mu\left[\tfrac{15}{16}\frac{a}{A} + \cdots + \text{func}\,(\mu, \theta)\right],$$

(3) When the values of α and e are substituted in (a) a factor μ^2 may be divided out and θ obtained as a power series in μ alone, vanishing with μ.

(4) By substitution of the value of θ thus found in the expressions for e and α, we obtain finally

$$\alpha = \mu\mathfrak{P}_1(\mu), \qquad e = \mu\mathfrak{P}_2(\mu), \qquad \theta = \mu\mathfrak{P}_3(\mu).$$

The preceding operations are known to be convergent for all values of α, e, θ, and μ which are sufficiently small. Hence for a given value of μ sufficiently small it is possible to determine the initial conditions (21) as power series in μ such that the solution of the differential equations (19) shall be periodic in τ with the period $2p\pi$.

Since $\mu\nu = N$, μ is determined by choosing the period of the solution in t

* Cf. POINCARÉ, *Les méthodes nouvelles de la mécanique céleste*, vol. 1 (1892), p. 87.

which is $2p\pi/\nu$. All distinct orbits are obtained by taking $p=1$, that is, every orbit is reëntrant after *one* revolution in the rotating plane. For an assigned period there is one and only one direct orbit, and one and only one retrograde orbit.

The analysis and results apply to the case when any number of finite bodies revolve about M in circles with the same angular velocity.

Case III. *When no uniform integral exists.* Suppose that M_1 and M_2 are not, in general, equal; also, that the eccentricity of the orbits of the finite bodies is not zero, and that the w-axis passes through M_1. These conditions are expressed by the relations

$$\epsilon \neq 0, \qquad \varpi_1 = 0, \qquad \varpi_2 = \frac{\pi}{3}. \tag{25}$$

When subject to the conditions (25), let the differential equations of motion be denoted by (7_{III}). In this case it is convenient to consider the equations

$$\begin{aligned} &\frac{d^2\rho}{d\tau^2} - \rho\left(\frac{dw}{d\tau} + \mu\frac{dU}{d\tau}\right)^2 + \frac{1}{\rho^2} = \mu^2 f_{\text{III}}, \\ &\rho\left(\frac{d^2w}{d\tau^2} + \mu\frac{d^2U}{d\tau^2}\right) + 2\frac{d\rho}{d\tau}\left(\frac{dw}{d\tau} + \mu\frac{dU}{d\tau}\right) = \mu^2 g_{\text{III}}, \end{aligned} \tag{26}$$

where $U = \mu/m \cdot V$ and $\mu^2 f_{\text{III}}$ and $\mu^2 g_{\text{III}}$ are obtained from $1/a \cdot \partial\Omega/\partial(a\rho)$ and $1/a^2\rho \cdot \partial\Omega/\partial w$ respectively, by writing μ^2 for the factor m^2 and making the substitution

$$\lambda_2 = \lambda\mu.$$

Equations (20) express the conditions that a solution shall be periodic with the period $2p\pi$. Consider the solution of equations (26) subject to the initial conditions (21), except that the fourth condition contains the term $-\mu \cdot dU/d\tau$ $(\tau = 0)$ instead of $-\mu$. The terms independent of μ are given by equations (22). The only term involving μ to the first degree is $-\mu/m \cdot U$, which occurs in the expression for w.

For the determination of the terms in μ^2 we have the equations

$$\begin{aligned} &\frac{d^2\rho_2}{d\tau^2} - 2\frac{dw_2}{d\tau} - 3\rho_2 = \frac{K\lambda_1}{R^3}\left[\tfrac{1}{2}(1+3\cos 2\tau) + \tfrac{3}{8}\frac{a}{AR}(3\cos\tau + 5\cos 3\tau) + \cdots\right], \\ &\frac{d^2w_2}{d\tau^2} + 2\frac{d\rho_2}{d\tau} = -\frac{K\lambda_1}{R^3}\left[\tfrac{3}{2}\sin 2\tau + \tfrac{3}{8}\frac{a}{AR}(\sin\tau + 5\sin 3\tau) + \cdots\right]. \end{aligned} \tag{27}$$

R is a power series in ϵ whose coefficients involve only cosines of multiples of $m\tau$. It follows that the right hand member of the first and second equations respectively have the forms

$$A_{0,0} + \sum_{g,h} A_{g,h}\cos(g+hm)\tau, \qquad \sum_{g,h} D_{g,h}\sin(g+hm)\tau,$$

where g and h are integers, g taking all positive values and zero, and h taking positive and zero values such that $hm < 1$. Since τ enters the differential equations explicitly, we have $m = q/p$, by equation (4). $A_{g,h}$ and $D_{g,h}$ are constants depending upon ϵ and a/A; they are infinite series which, for small values of the arguments, converge rapidly.

The second equation of the set (27) may be integrated. The result is *

$$\frac{dw_2}{d\tau} + 2\rho_2 = c_1 - \sum_{g,h} \frac{D_{g,h}}{g + hm} \cos(g + hm)\tau, \tag{28}$$

where c_1 is the constant of integration.

On eliminating $dw_2/d\tau$ between the first of equations (27) and equation (28), there results an equation of the form

$$\frac{d^2\rho_2}{d\tau^2} + \rho_2 = \bar{A}_{0,0} + \bar{A}_{1,0} \cos\tau + \cdots + \bar{A}_{g,h} \cos(g + hm)\tau + \cdots;$$

whence †

$$\rho_2 = c_2 \cos\tau + c_3 \sin\tau + \bar{A}_{0,0} + \tfrac{1}{2}\bar{A}_{1,0}\tau \sin\tau + \cdots + \frac{A_{g,h}}{1-(g+hm)^2} \cos(g+hm)\tau \cdots,$$

where c_2 and c_3 are constants of integration.

Substituting the value of ρ_2 in equation (28) we get

$$\frac{dw_2}{d\tau} = \bar{D}_{0,0} - 2c_3 \sin\tau - \bar{A}_{1,0}\tau \sin\tau + \cdots + \bar{D}_{g,h} \cos(g + hm)\tau + \cdots;$$

whence w_2 is obtained by quadrature. The constants of integration are determined by the initial conditions,

$$\rho_2 = \frac{d\rho_2}{d\tau} = w_2 = \frac{dw_2}{d\tau} = 0 \qquad (\tau = 0).$$

It is important to observe that all the terms of the solution of equations (27) have the period $2p\pi$, except the terms τ and $\tau\cos\tau$ which occur in w_2, and $\tau \sin\tau$ which occurs in ρ_2. Consequently when the conditions of periodicity (20) are applied, equations (a) and (d) will contain no term in μ^2 alone; (b) and (c) will each contain a term in μ^2 alone.

For the determination of the terms in $\mu^2\phi$ in the solution we have equations of the form

$$\begin{aligned} \frac{d^2\bar{\rho}}{d\tau^2} - 2\frac{d\bar{w}}{d\tau} - 3\bar{\rho} &= \sum_{g,h} G_{g,h} \sin(g + hm)\tau, \\ \frac{d^2\bar{w}}{d\tau^2} + 2\frac{d\bar{\rho}}{d\tau} &= H_{0,0} + \sum_{g,h} H_{g,h} \cos(g + hm)\tau. \end{aligned} \tag{29}$$

* By the definition of g, h, and m the denominator $g + hm$ cannot vanish.

† By the definition of g, h, and m the denominator $1 - (g + hm)^2$ cannot vanish.

The solution of the set (29) is of the form

$$\bar{\rho} = 2H_{0,0}\tau - \tfrac{1}{2}\bar{G}_{1,0}\tau\cos\tau + \cdots + \frac{\bar{G}_{g,h}}{1-(g+hm)^2}\sin(g+hm)\tau + \cdots,$$

$$\bar{w} = -\tfrac{3}{2}H_{0,0}\tau^2 + \bar{G}_{1,0}\tau\sin\tau + \cdots - \frac{\bar{H}_{g,h}}{g+hm}\cos(g+hm)\tau + \cdots.$$

The conditions that the solution of equations (26) under consideration shall be periodic with the period $2p\pi$ are

$$\begin{aligned}
0 &= -3p\pi\alpha e\theta + (4H_{0,0}p\pi - \bar{G}_{1,0}p\pi)\mu^2\phi + \cdots, && (a)\\
0 &= -3p\pi\alpha e + \bar{A}_{1,0}p\pi\mu^2 + \cdots, && (b)\\
0 &= -3p\pi\alpha + 2p\pi\mu + \cdots, && (c)\\
0 &= 6p\pi\alpha e\theta - (6H_{0,0}p\pi - 2\bar{G}_{1,0}p\pi)\mu^2\phi + \cdots. && (d)
\end{aligned} \tag{30}$$

Equations (30) have the following properties:

i. There are no terms independent of α and μ.

ii. The first power of μ does not occur in (a), (b), and (d).

iii. There are no terms involving ϕ which are independent of μ^2.

iv. There are no terms involving θ which are independent of e.

v. Equations (a), (b), and (d) contain no term in α independent of e.

vi. In equations (a) and (d) there are no terms of degree less than the third.

The knowledge of the preceding properties leads to the conclusion that α, e, θ, and ϕ may be determined uniquely as power series in μ, vanishing with μ and satisfying equations (30). This conclusion is reached by the following steps which are valid for sufficiently small values of α, e, θ, ϕ, and μ.

(1) Let equation (c) be solved for α as a power series in e, θ, ϕ, and μ (which contains μ as a factor by i), giving

$$\alpha = \tfrac{2}{3}\mu + \mu\mathfrak{P}_c(e, \theta, \phi, \mu). \tag{31}$$

Equation (31) may be used to eliminate α from equations (a), (b), and (d). Let the resulting equations be denoted by (a'), (b'), and (d') respectively.

(2) Then (b') contains a factor μ which may be divided out, leaving

$$0 = -2p\pi e + \bar{A}_{1,0}p\pi\mu + \cdots.$$

This equation contains no term independent of e and μ (by i, ii, and v), and contains a term of the first degree in e. Hence it may be solved for e as a power series in θ, ϕ, and μ, which vanishes with μ,

$$e = \tfrac{1}{2}\bar{A}_{1,0}\mu + \mu\mathfrak{P}_b(\theta, \phi, \mu). \tag{32}$$

Equation (32) may be used to eliminate e from equations (a') and (d'). Let the resulting equations be denoted by (a'') and (d'') respectively.

(3) Each of equations (a'') and (d'') contains μ^2 as a factor (by i, ii, iii, iv, v, vi), which may be divided out, leaving no term independent of θ, ϕ, and μ:

$$(33)\qquad \begin{aligned} 0 &= -p\pi\bar{A}_{1,0}\theta + (4H_{0,0}p\pi - \bar{G}_{1,0}p\pi)\phi + \cdots, && (a'')\\ 0 &= 2p\pi\bar{A}_{1,0}\theta - (6H_{0,0}p\pi - 2G_{1,0}p\pi)\phi + \cdots. && (b'')\end{aligned}$$

The jacobian with respect to θ and ϕ for $\theta = \phi = \mu = 0$ is $-2p^2\pi^2\bar{A}_{1,0}H_{0,0}$, which does not vanish identically in ϵ and a/A. Hence equations (33) may be solved uniquely for θ and ϕ as power series in μ, and the result used to eliminate θ and ϕ from equations (31) and (32). This yields the final solution of of equations (30),

$$(34)\qquad \alpha = \mu\mathfrak{P}_1(\mu), \qquad e = \mu\mathfrak{P}_2(\mu), \qquad \theta = \mu\mathfrak{P}_3(\mu), \qquad \phi = \mu\mathfrak{P}_4(\mu).$$

When these values are substituted in the solution of equations (26), the expressions for ρ and w become power series in μ alone, and the solution is periodic with the period $2p\pi$, since the conditions (30) are satisfied. The physical interpretation is the same as in the preceding cases.

For the purpose of constructing the solutions in applications, it is convenient to have the existence proof in the form given. It leaves open, however, one question which should be answered. That is, are there any values of ϵ different from zero for which the jacobian of equations (33) vanishes? The jacobian is a power series in ϵ and a/A, vanishing with ϵ, and it is not easy to discuss other values of ϵ for which the series might vanish. Such a discussion is made unnecessary by a slight variation in the details of the existence proof. The proof depends only upon the properties of the solution of the problem of two bodies and the non-periodic terms which enter in ρ_2, w_2, $\bar{\rho}$, and $\bar{w}$, and may be carried through without considering in equations (27) and (29) the infinite series in ϵ which represents $1/R$ and the infinite series in a/A. This is accomplished by picking out from f_{III} in (26) a single term of the type

$$\epsilon^p\left(\frac{a}{A}\right)^{q-1}\cos pm\tau\cos(q-1)w$$

and from g_{III} a single term of the type

$$\epsilon^p\left(\frac{a}{A}\right)^{q-1}\cos pm\tau\sin(q-1)w.$$

These terms are left unchanged, while in all other terms ϵ is replaced by $\sigma\mu$ and a/A by $\eta\mu$. The set of differential equations thus obtained may be treated by steps similar to those employed in the consideration of equations (26), and a set

of equations corresponding to (33) will be obtained. The jacobian of this set is $\epsilon^{2p}(a/A)^{2q-2}$, except for a constant factor, and can vanish only if $\epsilon = 0$.

Construction of the Solutions.

It has been demonstrated for each of the three sets of differential equations (11), (19), and (26) that, for a given value of μ sufficiently small, it is possible to determine the initial conditions uniquely as power series in μ such that the solution as power series in μ shall be periodic in τ with the period $2p\pi$. A method will now be given by which the solution to any desired number of terms may be conveniently constructed. It is not necessary to determine the initial conditions explicitly in advance and the computation involves only algebraic processes. It will be convenient for brevity in the treatment to consider first

Case III. *When no uniform integral exists.* Consider equations (26), and let the periodic solution, which is known to exist, be written

$$\begin{aligned} \rho - 1 &= \rho_1\mu + \rho_2\mu^2 + \rho_3\mu^3 + \cdots + \rho_i\mu^i + \cdots, \\ w - \tau &= w_1\mu + w_2\mu^2 + w_3\mu^3 + \cdots + w_i\mu^i + \cdots. \end{aligned} \tag{35}$$

Since the solution exists and is periodic for all values of μ sufficiently small, each coefficient is periodic with the required period.

Let the solution (35) be substituted into the differential equations (26) and the result arranged as power series in μ. The terms of the left-hand members have the following forms. (The accents indicate derivatives with respect to τ.)

$$U = \tau + \bar{U} = \tau + \frac{2}{m}\epsilon \sin m\tau + \frac{5}{4m}\epsilon^2 \sin 2m\tau + \cdots,$$

$$\frac{dU}{d\tau} = 1 + \bar{U}', \qquad \frac{d^2U}{d\tau^2} = \bar{U}'',$$

$$\frac{d^2\rho}{d\tau^2} = \rho_1''\mu + \rho_2''\mu^2 + \rho_3''\mu^3 + \cdots + \rho_i''\mu^i + \cdots,$$

$$\begin{aligned} \rho\left(\frac{dw}{d\tau} + \mu\frac{dU}{d\tau}\right)^2 &= 1 + [\rho_1 + 2(w_1' + 1 + \bar{U}')]\mu \\ &\quad + [\rho_2 + 2w_2' + 2\rho_1(w_1' + 1 + \bar{U}') + (w_1' + 1 + \bar{U}')^2]\mu^2 \\ &\quad + [\rho_3 + 2w_3' + 2(w_1' + 1 + \bar{U}')w_2' + 2\rho_1 w_2' + 2\rho_2(w_1' + 1 + \bar{U}') \\ &\quad + \rho_1(w_1 + 1 + \bar{U}')^2]\mu^3 + \cdots + [\rho_i + 2w_i' + 2(w_1' + 1 + \bar{U}')w_{i-1}' \\ &\quad + 2\rho_{i-1}(w_1' + 1 + \bar{U}') + 2\rho_1 w_{i-1}' + \cdots]\mu^i + \cdots, \end{aligned} \tag{36}$$

$$\begin{aligned} \frac{1}{\rho^2} &= 1 - 2\rho_1\mu - (2\rho_2 - 3\rho_1^2)\mu^2 - (2\rho_3 - 6\rho_1\rho_2 + 4\rho_1^3)\mu^3 \\ &\quad + \cdots + (2\rho_i - 6\rho_{i-1}\rho_1 + \cdots)\mu^i + \cdots, \end{aligned}$$

$$\rho\left(\frac{d^2w}{d\tau^2}+\mu\frac{d^2U}{d\tau^2}\right)=(w_1''+\bar{U}'')\mu+[w_2''+\rho_1(w_1''+\bar{U}'')]\mu^2$$
$$+[w_3''+w_2''\rho_1+\rho_2(w_1''+\bar{U}'')]\mu^3+\cdots$$
$$+[w_i''+w_{i-1}''\rho_1+\cdots+\rho_{i-1}(w_1''+\bar{U}'')]\mu^i+\cdots,$$

$$\frac{d\rho}{d\tau}\left(\frac{dw}{d\tau}+\mu\frac{dU}{d\tau}\right)=\rho_1'\mu+[\rho_2'+\rho_1'(w_1'+1+\bar{U}')]\mu^2$$
$$+[\rho_3'+\rho_2'(w_1'+1+\bar{U}')+\rho_1'w_2']\mu^3+\cdots$$
$$+[\rho_i'+\rho_{i-1}'(w_1'+1+\bar{U}')+\cdots+\rho_1'w_{i-1}']\mu^i+\cdots.$$

The equations hold identically in μ. Hence, equating the coefficients of the first power of μ, we have for the determination of ρ_1 and w_1 the following set of equations:

$$(37)\qquad \frac{d^2\rho_1}{d\tau^2}-2\frac{dw_1}{d\tau}-3\rho_1=2(1+\bar{U}'),\qquad \frac{d^2w_1}{d\tau^2}+2\frac{d\rho_1}{d\tau^2}=-\bar{U}''.$$

It follows that

$$(38)\qquad \begin{aligned}\rho_1&=2(1+c_1^{(1)})+c_2^{(1)}\cos\tau+c_3^{(1)}\sin\tau,\\ w_1&=c_4^{(1)}-(4+3c_1^{(1)})\tau-\bar{U}-2c_2^{(1)}\sin\tau+2c_3^{(1)}\cos\tau,\end{aligned}$$

where $c_1^{(1)}$, $c_2^{(1)}$, $c_3^{(1)}$, and $c_4^{(1)}$ are constants of integration. Since ρ_1 and w_1 are periodic, the coefficient of τ in w_1 must vanish. This condition determines the constant $c_1^{(1)}$, namely $c_1^{(1)}=-\frac{4}{3}$. The other constants of integration are so far arbitrary.

On equating the coefficients of the second power of μ, the following set of equations is obtained:

$$(39)\qquad \begin{aligned}\frac{d^2\rho_2}{d\tau^2}-2\frac{dw_2}{d\tau}-3\rho_2&=(w_1'+1+\bar{U}')^2+2\rho_1(w_1'+1+\bar{U}')-3\rho_1^2+f_{\mathrm{III},0},\\ \frac{d^2w_2}{d\tau^2}+2\frac{d\rho_2}{d\tau}&=-\rho_1(w_1''+\bar{U}'')-2\rho_1'(w_1'+1+\bar{U}')+g_{\mathrm{III},0},\end{aligned}$$

where $f_{\mathrm{III},0}$ and $g_{\mathrm{III},0}$ are obtained from f_{III} and g_{III} respectively by writing $\mu=0$, $w=\tau$, $\rho=1$. The right-hand members are known functions of τ and the equations have the form

$$(40)\qquad \begin{aligned}\frac{d^2\rho_2}{d\tau^2}-2\frac{dw_2}{d\tau}-3\rho_2&=A_{0,0}^{(2)}+(\tfrac{14}{3}c_2^{(1)}+A_{1,0}^{(2)})\cos\tau+\tfrac{14}{3}c_3^{(1)}\sin\tau\\ &\quad+\cdots+A_{g,h}^{(2)}\cos(g+hm)\tau+\cdots,\\ \frac{d^2w_2}{d\tau^2}+2\frac{d\rho_2}{d\tau}&=(\tfrac{10}{3}c_2^{(1)}+D_{1,0}^{(2)})\sin\tau-\tfrac{10}{3}c_3^{(1)}\cos\tau\\ &\quad+\cdots+D_{g,h}^{(2)}\sin(g+hm)\tau+\cdots.\end{aligned}$$

It is to be noted that $c_4^{(1)}$ does not occur in equations (40). Integrating the second equation we have

$$\frac{dw_2}{d\tau} + 2\rho_2 = c_1^{(2)} - (\tfrac{1}{3}c_2^{(1)} + D_{1,0}^{(2)})\cos\tau - \tfrac{1}{3}c_3^{(1)}\sin\tau + \cdots - \frac{D_{g,h}^{(2)}}{g+hm}\cos(g+hm)\tau + \cdots. \tag{41}$$

On eliminating $dw_2/d\tau$ from the first of equations (40) by means of equation (41), there results

$$\frac{d^2\rho_2}{d\tau^2} + \rho_2 = A_{0,0}^{(2)} + 2c_1^{(2)} + [-2c_2^{(1)} + A_{1,0}^{(2)} - 2D_{1,0}^{(2)}]\cos\tau - 2c_3^{(1)}\sin\tau + \cdots + \left(A_{g,h}^{(2)} - \frac{2D_{g,h}^{(2)}}{g+hm}\right)\cos(g+hm)\tau + \cdots. \tag{42}$$

In order that the solution of equation (42) shall contain no non-periodic term, the coefficients of $\cos\tau$ and $\sin\tau$ must vanish, whence $c_2^{(1)}$ and $c_3^{(1)}$ are determined by the conditions

$$2c_2^{(1)} = A_{1,0}^{(2)} - 2D_{1,0}^{(2)}, \qquad c_3^{(1)} = 0.$$

With these values of $c_2^{(1)}$ and $c_3^{(1)}$, the solution becomes

$$\rho_2 = A_{0,0}^{(2)} + 2c_1^{(2)} + c_2^{(2)}\cos\tau + c_3^{(2)}\sin\tau + \cdots + \alpha_{g,h}^{(2)}\cos(g+hm)\tau + \cdots,$$

where

$$\alpha_{g,h}^{(2)} = \frac{1}{1-(g+hm)^2}\left(A_{g,h}^{(2)} - \frac{2D_{g,h}^{(2)}}{g+hm}\right).$$

Substituting this value of ρ_2 in equation (41) and integrating, we obtain for w_2 a solution of the form

$$w_2 = c_4^{(2)} - (2A_{0,0}^{(2)} + 3c_1^{(2)})\tau - 2c_2^{(2)}\sin\tau + 2c_3^{(2)}\cos\tau + \cdots + \delta_{g,h}^{(2)}\sin(g+hm)\tau + \cdots,$$

where

$$\delta_{g,h}^{(2)} = -\frac{1}{g+hm}\left(\frac{D_{g,h}^{(2)}}{g+hm} + 2\alpha_{g,h}^{(2)}\right).$$

Since w_2 is periodic, $c_1^{(2)}$ is determined by the condition

$$2A_{0,0}^{(2)} + 3c_1^{(2)} = 0.$$

Of the eight constants of integration, which have been introduced in the first two steps, four ($c_1^{(1)}$, $c_2^{(1)}$, $c_3^{(1)}$, $c_1^{(2)}$) have been determined uniquely and four ($c_4^{(1)}$, $c_2^{(2)}$, $c_3^{(2)}$, $c_4^{(2)}$) are still arbitrary.

By equating the coefficients of the third power of μ the following set of equations is obtained:

$$\begin{aligned}
&\frac{d^2\rho_3}{d\tau^2} - 2\frac{dw_3}{d\tau} - 3\rho_3 = 2(w_1' + 1 + \bar{U}' + \rho_1)w_2' + 2\rho_2(w_1' + 1 + \bar{U}') \\
&\qquad + \rho_1(w_1' + 1 + \bar{U}')^2 - 6\rho_1\rho_2 + 4\rho_1^3 + f_{\mathrm{III},1}, \\
&\frac{d^2w_3}{d\tau^2} + 2\frac{d\rho_3}{d\tau} = -\rho_1 w_2'' - (w_1'' + \bar{U}'')\rho_2 - 2\rho_2'(w_1' + 1 + \bar{U}') - 2\rho_1' w_2' + g_{\mathrm{III},1},
\end{aligned} \tag{43}$$

where $f_{\mathrm{III},1}$ and $g_{\mathrm{III},1}$ denote the coefficients of μ in f_{III} and g_{III} respectively. Equations (43) do not contain $c_4^{(2)}$. The right-hand members contain $c_4^{(1)}$ linearly. The equations have the form

$$\begin{aligned}\frac{d^2\rho_3}{d\tau^2} - 2\frac{dw_3}{d\tau} - 3\rho_3 &= A^{(3)}_{0,0} + (\tfrac{1}{3}^4 c_2^{(2)} + A^{(3)}_{1,0})\cos\tau + (\tfrac{1}{3}^4 c_3^{(2)} + B^{(3)}_{1,0})\sin\tau \\ &\quad + \cdots + A^{(3)}_{g,h}\cos(g+hm)\tau + B^{(3)}_{g,h}\sin(g+hm)\tau + \cdots, \\ \frac{d^2w_3}{d\tau^2} + 2\frac{d\rho_3}{d\tau} &= C^{(3)}_{0,0} + (\tfrac{1}{3}^0 c_2^{(2)} + D^{(3)}_{1,0})\sin\tau + (-\tfrac{1}{3}^0 c_3^{(2)} + C^{(3)}_{1,0})\cos\tau \\ &\quad + \cdots D^{(3)}_{g,h}\sin(g+hm)\tau + C^{(3)}_{g,h}\cos(g+hm)\tau + \cdots.\end{aligned} \tag{44}$$

Here $C^{(3)}_{0,0}$ involves $c_4^{(1)}$ linearly. To avoid non-periodic terms in the solution, $C^{(3)}_{0,0}$ must vanish, and this condition determines $c_4^{(1)}$ uniquely, if the coefficient of $c_4^{(1)}$ does not vanish. The coefficient is a power series in ϵ and a/A which vanishes with ϵ but not identically in ϵ. The question concerning special values of ϵ for which it might vanish is the same as the question concerning the vanishing of the jacobian of equations (33) and is answered in the same way.

The treatment of equations (44) proceeds by steps similar to those employed in the solution of equations (40). Four new constants of integration are introduced, namely, $c_1^{(3)}$, $c_2^{(3)}$, $c_3^{(3)}$, $c_4^{(3)}$; while $c_2^{(2)}$, $c_3^{(2)}$, and $c_1^{(3)}$ are uniquely determined by the conditions:

$$2c_2^{(2)} = A^{(3)}_{1,0} - 2D^{(3)}_{1,0}, \qquad 2c_3^{(2)} = B^{(3)}_{1,0} + 2C^{(3)}_{1,0}, \qquad 3c_1^{(3)} = -2A^{(3)}_{0,0}.$$

So far $c_4^{(2)}$, $c_2^{(3)}$, $c_3^{(3)}$, and $c_4^{(3)}$ are arbitrary.

It may be established by complete induction that the above process can be carried as far as is desired. Suppose $\rho_1, w_1;\ \rho_2, w_2;\ \cdots \rho_{i-1}, w_{i-1}$ have been determined by this process. The expressions have the following form

$$\begin{aligned}\rho_l &= \alpha^{(l)}_{0,0} + \cdots + \alpha^{(l)}_{g,h}\cos(g+hm)\tau + \beta^{(l)}_{g,h}\sin(g+hm)\tau + \cdots, \\ w_l &= \gamma^{(l)}_{0,0} + \cdots + \delta^{(l)}_{g,h}\sin(g+hm)\tau + \gamma^{(l)}_{g,h}\cos(g+hm)\tau + \cdots,\end{aligned} \qquad (l = 1, 2, \cdots, i-3).$$

$$\begin{aligned}\rho_{i-2} &= \alpha^{(i-2)}_{0,0} + \cdots + \alpha^{(i-2)}_{g,h}\cos(g+hm)\tau + \beta^{(i-2)}_{g,h}\sin(g+hm)\tau + \cdots, \\ w_{i-2} &= c_4^{(i-2)} + \cdots + \delta^{(i-2)}_{g,h}\sin(g+hm)\tau + \gamma^{(i-2)}_{g,h}\cos(g+hm)\tau + \cdots.\end{aligned}$$

$$\begin{aligned}\rho_{i-1} &= \alpha^{(i-1)}_{0,0} + c_2^{(i-1)}\cos\tau + c_3^{(i-1)}\sin\tau + \cdots, \\ w_{i-1} &= c_4^{(i-1)} - 2c_2^{(i-1)}\sin\tau + 2c_3^{(i-1)}\cos\tau + \cdots.\end{aligned}$$

The equations for the determination of ρ_i and w_i are

$$\begin{aligned}\frac{d^2\rho_i}{d\tau^2} - 2\frac{dw_i}{d\tau} - 3\rho_i &= A^{(i)}_{0,0} + (\tfrac{1}{3}^4 c_2^{(i-1)} + A^{(i)}_{1,0})\cos\tau + (\tfrac{1}{3}^4 c_3^{(i-1)} + B^{(i)}_{1,0})\sin\tau \\ &\quad + \cdots + A^{(i)}_{g,h}\cos(g+hm)\tau + B^{(i)}_{g,h}\sin(g+hm)\tau + \cdots, \\ \frac{d^2w_i}{d\tau^2} + 2\frac{d\rho_i}{d\tau} &= C^{(i)}_{0,0} + (\tfrac{1}{3}^0 c_2^{(i-1)} + D^{(i)}_{1,0})\sin\tau + (-\tfrac{1}{3}^0 c_3^{(i-1)} + C^{(i)}_{1,0})\cos\tau \\ &\quad + \cdots + D^{(i)}_{g,h}\sin(g+hm)\tau + C^{(i)}_{g,h}\cos(g+hm)\sin\tau + \cdots.\end{aligned} \tag{45}$$

Equations (45) do not contain $c_4^{(i-1)}$. The coefficient of $c_4^{(i-2)}$ in $C_{0,0}^{(i)}$ is the same as the coefficient of $c_4^{(1)}$ in $C_{0,0}^{(3)}$. Equations (45) are solved by the steps employed for the solution of equations (40). During the process four constants of integration are introduced, namely, $c_1^{(i)}, c_2^{(i)}, c_3^{(i)}, c_4^{(i)}$, and four are uniquely determined by the following conditions, the first of which determines $c_4^{(i-2)}$;

$$
\begin{aligned}
&C_{0,0}^{(i)} = 0, && 2c_2^{(i-1)} = A_{1,0}^{(i)} - 2D_{1,0}^{(i)},\\
&2c_3^{(i-1)} = B_{1,0}^{(i)} + 2C_{1,0}^{(i)}, && 3c_1^{(i)} = -2A_{0,0}^{(i)}.
\end{aligned}
\tag{46}
$$

The solution of equations (45) is

$$
\begin{aligned}
\rho_i &= \alpha_{0,0}^{(i)} + c_2^{(i)}\cos\tau + c_3^{(i)}\sin\tau + \cdots\\
&\qquad + \alpha_{g,h}^{(i)}\cos(g+hm)\tau + \beta_{g,h}^{(i)}\sin(g+hm)\tau + \cdots,\\
w_i &= c_4^{(i)} + \delta_{1,0}^{(i)}\sin\tau + \gamma_{1,0}^{(i)}\cos\tau + \cdots\\
&\qquad + \delta_{g,h}^{(i)}\sin(g+hm)\tau + \gamma_{g,h}^{(i)}\cos(g+hm)\tau + \cdots,
\end{aligned}
\tag{47}
$$

where the coefficients are given by the formulas:

$$
\begin{aligned}
&\alpha_{0,0}^{(i)} = -\tfrac{1}{3}A_{0,0}^{(i)},\\
&\alpha_{g,h}^{(i)} = \frac{1}{1-(g+hm)^2}\left(A_{g,h}^{(i)} - \frac{2D_{g,h}^{(i)}}{g+hm}\right),\\
&\beta_{g,h}^{(i)} = \frac{1}{1-(g+hm)^2}\left(B_{g,h}^{(i)} + \frac{2C_{g,h}^{(i)}}{g+hm}\right),\\
&\delta_{1,0}^{(i)} = -2c_2^{(i)} - \tfrac{5}{3}A_{1,0}^{(i)} + \tfrac{7}{3}D_{1,0}^{(i)},\\
&\delta_{g,h}^{(i)} = -\frac{1}{g+hm}\left(\frac{D_{g,h}^{(i)}}{g+hm} + 2\alpha_{g,h}^{(i)}\right),\\
&\gamma_{1,0}^{(i)} = 2c_3^{(i)} + \tfrac{5}{3}B_{1,0}^{(i)} + \tfrac{7}{3}C_{1,0}^{(i)},\\
&\gamma_{g,h}^{(i)} = -\frac{1}{g+hm}\left(\frac{C_{g,h}^{(i)}}{g+hm} - 2\beta_{g,h}^{(i)}\right).
\end{aligned}
\tag{48}
$$

This completes the proof of the statement that the solution may be constructed to any desired degree of accuracy. Of the four constants of integration which occur in the calculation of the coefficients of μ^i, one, $c_1^{(i)}$, is determined in the step in which it enters; two more, $c_2^{(i)}$ and $c_3^{(i)}$, are determined in the next following step; while the fourth, $c_4^{(i)}$, is determined in the second following step.

Case II. *When a uniform integral exists.* The method of constructing the solution in this case differs from the preceding in one particular. The general steps are the same and formulas (48) are applicable. The difference occurs in the determination of the constant $c_4^{(i-2)}$. In the proof of the existence of a

periodic solution of equations (19) it was shown that the origin of time is arbitrary, which is equivalent to the statement that at $\tau = \tau_0$ the value $w = w_0$ may be assumed arbitrarily. It follows that $C^{(i)}_{0,0} = 0$ and does not determine $c_4^{(i-2)}$. This constant is determined by assuming an initial condition, for example, $w = 0$ at $\tau = 0$. Then $c_4^{(i-2)}$ is determined in the step in which it enters by the condition $w_{i-2} = 0$ at $\tau = 0$. When the step for computing the coefficients of μ^i is completed, there remain two constants of integration, $c_2^{(i)}$ and $c_3^{(i)}$ which have not been determined. These constants are determined in the next following step.

Case I. *Symmetrical solutions.* In this case there are two arbitrary initial conditions and the method of constructing the solution differs from that of case III in two particulars. The differential equations are (11). Consider the first step in the construction [equations (37) and (38)]. We determine $c_3^{(1)}$ and $c_4^{(1)}$ respectively by the conditions

$$\frac{d\rho_1}{d\tau} = 0, \qquad w_1 = 0 \qquad (\tau = 0).$$

Therefore $c_3^{(1)} = c_4^{(1)} = 0$. And $c_1^{(1)}$ is determined by the condition that w_1 shall be periodic while $c_2^{(1)}$ is not determined until the next step.

This process is applicable to all the following steps. The differential equations (11) have a particular form which admits a symmetrical solution, and it may be established by complete induction that the equations for the determination of ρ_i and w_i have the form

$$\begin{aligned} &\frac{d^2\rho_i}{d\tau^2} - 2\frac{dw_i}{d\tau} - 3\rho_i = A^{(i)}_{0,0} + (\tfrac{1}{3}A c_2^{(i-1)} + A^{(i)}_{1,0})\cos\tau + \cdots + A^{(i)}_{g,h}\cos(g+hm)\tau + \cdots, \\ &\frac{d^2w_i}{d\tau^2} + 2\frac{d\rho_i}{d\tau} = (\tfrac{1}{3}c_2^{(i-1)} + D^{(i)}_{1,0})\sin\tau + \cdots + D^{(i)}_{g,h}\sin(g+hm)\tau + \cdots. \end{aligned} \tag{49}$$

where $c_2^{(i-1)}$ is determined by the condition

$$2c_2^{(i-1)} = A^{(i)}_{1,0} - 2D^{(i)}_{1,0}.$$

The solution of (49) is

$$\rho_i = \alpha^{(i)}_{0,0} + c_1^{(i)}\cos\tau + \cdots + \alpha^{(i)}_{g,h}\cos(g+hm)\tau + \cdots,$$

$$w_i = \delta^{(i)}_{1,0}\sin\tau + \cdots + \delta^{(i)}_{g,h}\sin(g+hm)\tau + \cdots,$$

where

$$\alpha^{(i)}_{0,0} = -\tfrac{1}{3}A^{(i)}_{0,0},$$

$$\alpha^{(i)}_{g,h} = \frac{1}{1-(g+hm)^2}\left(A^{(i)}_{g,h} - \frac{2D^{(i)}_{g,h}}{g+hm}\right),$$

$$\delta^{(i)}_{1,0} = -2c_2^{(i)} - \tfrac{5}{3}A^{(i)}_{1,0} + \tfrac{7}{3}D^{(i)}_{1,0},$$

$$\delta^{(i)}_{g,h} = -\frac{1}{g+hm}\left(\frac{D^{(i)}_{g,h}}{g+hm} + 2\alpha^{(i)}_{g,h}\right).$$

Numerical Examples.

As illustrations of the preceding analysis three numerical examples will be treated briefly. It has not been shown that the processes are valid for the numerical values which are employed and which have been selected for convenience in graphical representation. It is probable that the series are convergent, although it has not been found possible to determine the true radii of convergence.

In each case the mass of M is taken as the unit of mass, and M_1, of mass 10, is supposed to move about M in a circle whose radius is taken as the unit of distance. The unit of time is chosen so that the angular velocity N of M_1 is unity. The orbits considered are those in which the particle revolves about M. The period is chosen so that $\nu = 5$, whence $\mu = 0.2$.

In the first example M_2 of mass 5 is supposed to revolve about M in a circle so that the three finite masses remain always in a straight line in the order $M_2 M M_1$. It follows from the solution of the quintic equation of LAGRANGE that the distance $M_2 M$ is $0.77172\cdots$. The orbit of the particle is symmetrical with respect to the line joining the bodies.

In the second example M_2 of mass 5 is supposed to revolve about M in a circle so that the three bodies remain always at the vertices of an equilateral triangle. The differential equations admit the integral of JACOBI, but there is no symmetry theorem.

In the third case it is supposed that M_2 of unit mass revolves about M_1 in a circle of radius A_2. The orbit of the particle is symmetrical with respect to the radius vector of M_1 and crosses this line when the finite bodies are in conjunction.

The differential equations of motion. With reference to M as origin and an axis passing always through M_1 let the coördinates of M_1, M_2, and P be respectively $(1, 0)$, (R_2, W_2), and (r, w). The equations of motion, see equations (7), of the particle are

$$\frac{d^2\rho}{d\tau^2} - \rho\left(\frac{dw}{d\tau} + \mu\right)^2 + \frac{1}{\rho^2} = k^2 M_1 \mu^2 \rho \left[\tfrac{1}{2}\{1 + 3\cos 2w\} + \tfrac{3}{8} a\rho\{3\cos w + 5\cos 3w\}\right.$$
$$\left. + \tfrac{1}{16} a^2\rho^2\{9 + 20\cos 2w + 35\cos 4w\} + \cdots\right]$$
$$+ \frac{k^2 M_2}{R_2^3}\mu^2\rho\left[\tfrac{1}{2}\{1 + 3\cos 2(w - W_2)\} + \tfrac{3}{8}\frac{a}{R^2}\rho\{3\cos(w - W_2) + 5\cos 3(w - W_2)\}\right.$$
$$\left. + \tfrac{1}{16}\left(\frac{a}{R_2}\right)^2\rho^2\{9 + 20\cos 2(w - W_2) + 35\cos 4(w - W_2)\} + \cdots\right],$$

$$\rho\frac{d^2 w}{d\tau^2} + 2\frac{d\rho}{d\tau}\left(\frac{dw}{d\tau} + \mu\right) = -k^2 M_1 \mu^2 \rho\left[\tfrac{3}{2}\sin 2w + \tfrac{3}{8} a\rho\{\sin w + 5\sin 3w\}\right.$$
$$\left. + \tfrac{5}{16} a^2\rho^2\{2\sin 2w + 7\sin 4w\} + \cdots\right]$$
$$- \frac{k^2 M_2}{R_2^3}\mu^2\rho\left[\tfrac{3}{2}\sin 2(w - W_2) + \tfrac{3}{8}\frac{a}{R_2}\rho\{\sin(w - W_2) + 5\sin 3(w - W_2)\}\right.$$
$$\left. + \tfrac{5}{16}\left(\frac{a}{R_2}\right)^2\rho^2\{2\sin 2(w - W_2) + 7\sin 4(w - W_2)\} + \cdots\right],$$

where a is given by the relation $\nu^2 a^3 = k^2$. The numerical value of k^2 depends upon the masses and orbits of the finite bodies and must be determined separately for each of the three examples.

Example 1. In this case $R_2 = 0.77172$ and $W_2 = \pi$. Now k^2 is given by the relation

$$N^2 = \frac{M_1 + M + M_2}{M + M_2 + M_2 R_2}\left(M + \frac{M_2}{(1 + R_2^2)}\right)k^2,$$

whence

$$k^2 = 0.23763.$$

It follows that

$$k^2 M_1 = 2.37630, \qquad \frac{k^2 M_2}{R_2^3} = 2.58518,$$

$$a = 1.05914\mu, \qquad \frac{a}{R_2} = 1.37222\mu.$$

The equations of motion become

$$\begin{aligned}
\frac{d^2\rho}{d\tau^2} - \rho\left(\frac{dw}{d\tau} + \mu\right)^2 + \frac{1}{\rho^2} &= (2.48074 + 7.44222\cos 2w)\rho\mu^2 \\
&\quad - (1.15938\cos w + 1.93230\cos 3w)\rho^2\mu^3 \\
&\quad + (4.23765 + 9.41700\cos 2w + 16.47975\cos 4w)\rho^3\mu^4 + \cdots, \\
\rho\frac{d^2 w}{d\tau^2} + 2\frac{d\rho}{d\tau}\left(\frac{dw}{d\tau} + \mu\right) &= -(7.44222\sin 2w)\rho\mu^2 \\
&\quad + (0.38646\sin w + 1.93230\sin 3w)\rho^2\mu^3 \\
&\quad - (4.70850\sin 2w + 16.47975\sin 4w)\rho^3\mu^4 + \cdots.
\end{aligned} \tag{50}$$

The periodic solution of equations (50) is

$$\begin{aligned}
\rho &= 1 - \tfrac{2}{3}\mu - (0.27136 + 0.96615\cos\tau + 4.96148\cos 2\tau)\mu^2 \\
&\quad + (0.62584 - 19.13740\cos\tau - 2.47963\cos 2\tau + 0.40256\cos 3\tau)\mu^3 + \cdots, \\
w &= \tau + (1.93230\sin\tau + 6.82204\sin 2\tau)\mu^2 \\
&\quad + (41.10885\sin\tau + 10.74793\sin 2\tau - 0.48307\sin 3\tau)\mu^3 + \cdots.
\end{aligned} \tag{51}$$

Example 2. In this case $R_2 = 1$ and $W_2 = \pi/3$. Here k^2 is given by the relation

$$N^2 = k^2(M + M_1 + M_2),$$

whence

$$k^2 = 0.06250.$$

It follows that

$$k^2 M_1 = 0.62500, \qquad k^2 M_2 = 1.56250\mu, \qquad a = 0.67860\mu.$$

The equations of motion become

$$\begin{aligned}\frac{d^2\rho}{d\tau^2} - \rho\left(\frac{dw}{d\tau}+\mu\right)^2 + \frac{1}{\rho^2} &= (0.31250 + 0.93750\cos 2w)\rho\mu^2 \\ &+ (0.78125 - 1.17188\cos 2w + 2.02977\sin 2w)\rho\mu^3 \\ &+ (0.47714\cos w + 0.79523\cos 3w)\rho^2\mu^3 \\ &+ (0.59643\cos w + 1.03308\sin w - 1.98808\cos 3w)\rho^2\mu^4 \\ &+ (0.16188 + 0.35974\cos 2w + 0.62954\cos 4w)\rho^3\mu^4 + \cdots, \\ \rho\frac{d^2w}{d\tau^2} + 2\frac{d\rho}{d\tau}\left(\frac{dw}{d\tau}+\mu\right) &= -(0.93750\sin 2w)\rho\mu^2 \\ &+ (1.17188\sin 2w + 2.02977\cos 2w)\rho\mu^3 \\ &- (0.15905\sin w + 0.79523\sin 3w)\rho^2\mu^3 \\ &+ (-0.19881\sin w + 0.34436\cos w + 1.98808\sin 3w)\rho^2\mu^4 \\ &- (0.17987\sin 2w + 0.62954\sin 4w)\rho^3\mu^4 + \cdots.\end{aligned} \tag{52}$$

The periodic solution of equations (52) is

$$\begin{aligned}\rho &= 1 - \tfrac{2}{3}\mu + (0.45139 + 0.39762\cos\tau - 0.62500\cos 2\tau)\mu^2 \\ &+ (-0.47647 + 1.04542\cos\tau + 0.86090\sin\tau + 0.46875\cos 2\tau \\ &- 1.35318\sin 2\tau - 0.16567\cos 3\tau)\mu^3 + \cdots, \\ w &= \tau + (-0.79524\sin\tau + 0.85938\sin 2\tau)\mu^2 \\ &+ (0.13882 - 3.25720\sin\tau + 1.72180\cos\tau + 0.27995\sin 2\tau \\ &- 1.86062\cos 2\tau + 0.19881\sin 3\tau)\mu^3 + \cdots.\end{aligned} \tag{53}$$

Example 3. In this case M_2 revolves about M_1 in a circle of radius A_2 with an angular velocity N_2. Let N_2 be determined by the relation

$$N_2 - N = 2\nu.$$

A_2, R_2, and W_2 are determined by the relations

$$N_2^2 A_2^3 = k^2(M_1 + M_2),$$

$$R_2 = \sqrt{1 + A_2^2 + 2A_2\cos 2\nu t} = \sqrt{1 + A_2^2 + 2A_2\cos 2\tau},$$

$$\sin W_2 = \frac{A_2\sin 2\tau}{R_2},$$

$$\cos W_2 = \frac{R_2^2 + 1 - A_2^2}{2R_2}.$$

Here k^2 is given by the relation

$$N^2 = k^2(M + M_1),$$

whence

$$k^2 = 0.09091.$$

It follows that

$$k^2 M_1 = 0.90909, \qquad k^2 M_2 = 0.09091,$$

$$A_2 = 1.01200\mu, \qquad a = 0.76910\mu.$$

The differential equations become

$$\begin{aligned}
\frac{d^2\rho}{d\tau^2} - \rho\left(\frac{dw}{d\tau} + \mu\right)^2 + \frac{1}{\rho^2} &= (0.50000 + 1.50000\cos 2w)\rho\mu^2 \\
&+ (0.86523\cos w + 1.44205\cos 3w)\rho^2\mu^3 \\
&+ (0.33273 + 0.73940\cos 2w + 1.29395\cos 4w)\rho^3\mu^4 \\
&+ (0.27600\sin 2\tau\sin 2w - 0.13800\cos 2\tau - 0.41400\cos 2\tau\cos 2w)\rho\mu^3 \\
&+ (0.10474 + 0.17456\cos 4\tau + 0.31422\cos 2w + 0.10471\cos 4\tau\cos 2w \\
&\qquad - 0.55864\sin 4\tau\sin 2w)\rho\mu^4 \\
&+ (0.07960\sin 2\tau\sin w - 0.31840\cos 2\tau\cos w - 0.53068\cos 2\tau\cos 3w \\
&\qquad + 0.39801\sin 2\tau\sin 3w)\rho^2\mu^4 + \cdots, \qquad (54)
\end{aligned}$$

$$\begin{aligned}
\rho\frac{d^2w}{d\tau^2} + 2\frac{d\rho}{d\tau}\left(\frac{dw}{d\tau} + \mu\right) &= -(1.50000\sin 2w)\rho\mu^2 \\
&\quad - (0.28841\sin w + 1.44205\sin 3w)\rho^2\mu^3 \\
&\quad - (0.36970\sin 2w + 1.29395\sin 4w)\rho^3\mu^4 \\
&+ (0.27600\sin 2\tau\cos 2w + 0.41400\cos 2\tau\sin 2w)\rho\mu^3 \\
&- (0.31422\sin 2w + 0.10471\cos 4\tau\sin 2w + 0.55864\sin 4\tau\cos 2w)\rho\mu^4 \\
&+ (0.02653\sin 2\tau\cos w + 0.10613\cos 2\tau\sin w + 0.39801\sin 2\tau\cos 3w \\
&\qquad + 0.53068\cos 2\tau\sin 3w)\rho^2\mu^4 + \cdots.
\end{aligned}$$

The periodic solution of equations (54) is

$$\begin{aligned}
\rho &= 1 - \tfrac{2}{3}\mu + (0.38889 + 0.72102\cos\tau - \cos 2\tau)\mu^2 \\
&\quad + (-0.02616 + 2.09168\cos\tau - 0.45400\cos 2\tau - 0.30043\cos 3\tau \\
&\qquad + 0.03450\cos 4\tau)\mu^3 + \cdots, \qquad (55) \\
w &= \tau + (-1.44204\sin\tau + 1.37500\sin 2\tau)\mu^2 \\
&\quad + (-6.29838\sin\tau + 2.12066\sin 2\tau + 0.36051\sin 3\tau \\
&\qquad - 0.03881\sin 4\tau)\mu^3 + \cdots.
\end{aligned}$$

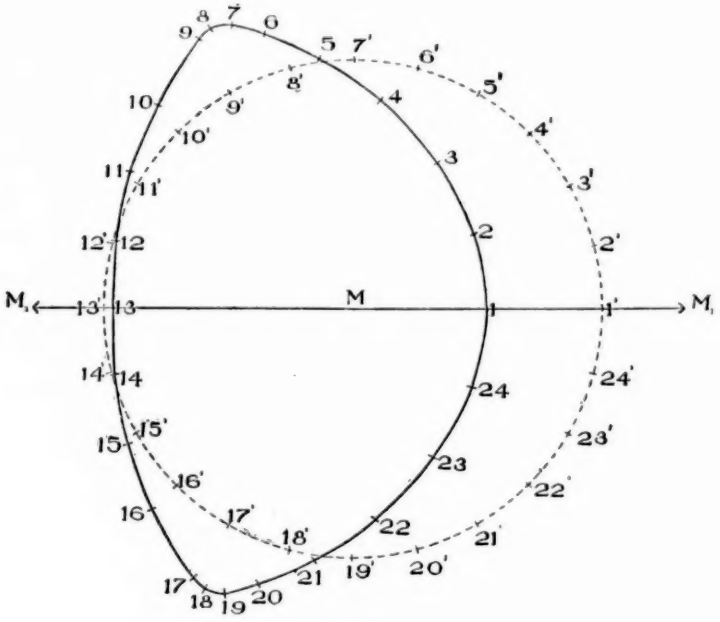

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FIG. 2.

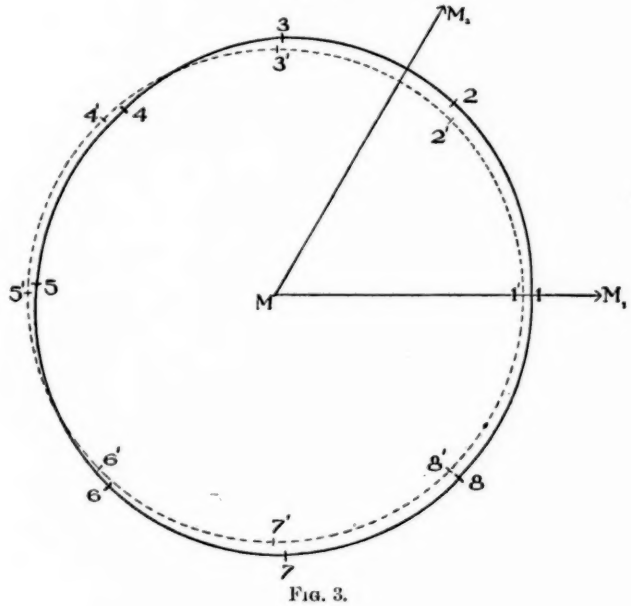
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FIG. 3.

The orbits represented by the solutions (51), (53), and (55) are illustrated by figures 2, 3, and 4 respectively. The comparison circles in the figures are not the circular orbits which have been called the *undisturbed orbits*. The undisturbed orbits are referred to fixed axes while the drawings are made with reference to

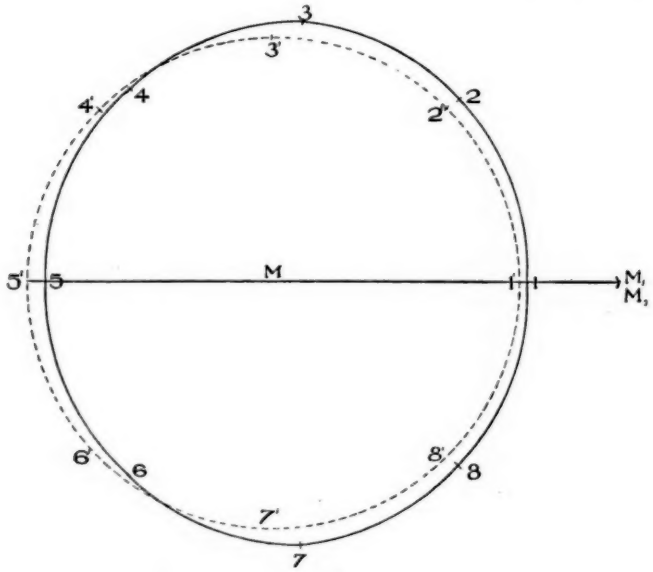

FIG. 4.

rotating axes. The comparison circles represent orbits in which the particle would make a complete revolution with respect to the rotating axes during the period. In figure 2 the points which are numbered 1, 2, $\cdots$ represent the positions of the particle in the periodic orbit at intervals in τ of $\pi/12$. The corresponding positions in the comparison circle are indicated by the numbers $1'$, $2'$, $\cdots$. In figures 3 and 4 the intervals are $\pi/4$.

A PROOF OF SOME THEOREMS ON POINTWISE DISCONTINUOUS FUNCTIONS*

BY

EDWARD B. VAN VLECK

BAIRE in his important thesis † has given a number of interesting theorems concerning pointwise discontinuous functions. For their demonstration he employs the concept of semi- (i. e., upper or lower) continuity. ‡ However necessary the concept may be for subsequent portions of his investigation, its introduction is not needed for the particular theorems referred to. When the unnecessary element is removed, the principles of their demonstration, although very different in form, become equivalent to those used by OSGOOD § in the derivation of his theorem relating to the convergence of series of continuous functions. (Cf. § 5 below.)

The object of the following paper is, in part, to establish the theorems without going so far as to introduce the notion of upper continuity, thus reducing the proof to somewhat lower terms. It is found that BAIRE has restricted $f(x, y)$ more than is necessary in the following theorem:

If $f(x, y)$ is continuous in x and y separately and has a partial derivative $\partial f/\partial x$ at every point of

$$T: \quad a \leqq x \leqq b, \qquad c \leqq y \leqq d,$$

this derivative, regarded as a function of (x, y), is at most only pointwise discontinuous.

In fact, it suffices here to add to the hypothesis of the existence of $\partial f/\partial x$ the condition that $f(x, y)$ shall be continuous in y upon a set of parallels to the Y-axis which are everywhere dense in T (see theorem II). Other extensions of BAIRE's results will be noted further on (cf. theorems III and V).

It is believed that in the analysis given here the necessary principles have been arranged and combined in so simple a form as to permit quick and easy

* Presented to the Society October 29, 1904, with a supplementary note April 29, 1905. Received for publication January 16, 1907.

† Annali di Matematica, ser. 3, vol. 3 (1899).

‡ Loc. cit., p. 6, or see SCHÖNFLIES' admirable *Bericht über die Mengenlehre*, Jahresbericht der deutschen Mathematiker-Vereinigung, vol. 8 (1900), p. 141. Reference to this report will be indicated hereafter by the word *Bericht*.

§ American Journal of Mathematics, vol. 19 (1897), p. 155.

application to other problems of like character which may arise. As instances of such application I will cite the two following theorems, which I think are new and which constitute a second object of this paper:

THEOREM VI.* *If throughout the field*

$$T: \quad a \leqq x \leqq b, \qquad c \leqq y \leqq d,$$

the function $f(x, y)$ is continuous in x when y is constant and in y when x is constant, then

$$F(x) \equiv \int_c^d f(x, y)\, dy$$

is at most a pointwise discontinuous function in the interval $a \leqq x \leqq b$.

THEOREM VII. *If at every point of T the integrand possesses a partial derivative $f'_x(x, y)$ which is continuous in the variables x and y considered separately, the integral $F(x)$ will have a derivative in some set of points which is everywhere dense in (a, b), and the value of the derivative at the points here found will be obtained by differentiating with respect to the parameter under the integral sign; i. e.,*

$$F'(x) = \int_c^d f'_x(x, y)\, dy.$$

Furthermore, these points are points of continuity for

$$\int_c^d f'_x(x, y)\, dy.$$

Concerning the former of the two theorems it may be remarked that it has been known previously † that when $|f(x, y)|$ has an upper limit in T the integral $F(x)$ is continuous in (ab). The cases of pointwise discontinuity therefore arise only when there is no upper limit for $|f(x, y)|$.

§ 1. *The preliminary principles.*

We shall first demonstrate the following very simple

Fundamental Principle: ‡ *If to every point p_i of a continuum (or of a perfect set of points)* § *a positive number δ_i is assigned in any manner, then there is some portion of the continuum (some perfect subset) where the points q_i*

* The two theorems are numbered with reference to the place in which they are proved later on.

† ARZELA, Atti della R. Accademia dei Lincei, ser. 4, vol. 1 (1884–5), p. 537. It also can be deduced immediately from OSGOOD's theorem 5, loc. cit., p. 182.

‡ A special case in which this principle is used incidentally will be found in the work of OSGOOD, Mathematische Annalen, vol. 53 (1900), p. 463. Other instances of the use of the principle can doubtless be found.

§ No restriction is here placed upon the perfect point-set. It may consist of a set of points nowhere dense in the space in which it is contained. The continuum may be either closed or open.

which have $\delta_i > D$ are everywhere dense (dense with respect to the subset), provided D is taken sufficiently small.*

Let us suppose, if possible, that this is not true. Denote by $\zeta_1, \zeta_2, \zeta_3, \cdots$ any sequence of decreasing positive numbers having 0 as their limit. Further let Q_i denote the set of points q_i for which $\delta_i > \zeta_i$. Clearly Q_i will be contained in Q_{i+1}. If, now, the principle stated is false, then there is some portion of the continuum (some perfect subset) which contains no points of Q_1. Within this portion (subset) there must be another part (perfect set) which is free from the points of Q_2, and so on. The limit of this series of sub-portions (sub-sets), contained each within the preceding, is either a point † to which belongs a non-negative number smaller than any ζ_i, or a set of such points. But 0 is the only number having this property. This, however, contradicts the hypothesis that to every point there corresponds a *positive* number. The fundamental principle is therefore established.

By placing $\delta_i = 1/\delta_i'$ and $D = 1/D'$ the principle may be put also into the following form: If to every point of the continuum a positive number δ_i' is assigned in any way, the points q_i for which $\delta_i' < D'$ must be everywhere dense in some portion of the continuum, provided D' is sufficiently large.

Consider next the set S of points s_i for which the assigned numbers δ_i fall between D and $1/D$. It will be seen at once that there must be some portion of the continuum (some perfect subset) in which the points s_i are everywhere dense, provided, of course, that D is sufficiently small. Let the interval $(D, 1/D)$ be divided into any number of equal parts,

$$\epsilon = \frac{1/D - D}{n},$$

and denote by $S_1, S_2, \cdots, S_n$ the subsets of S for which the values of δ_i fall successively in the intervals $(D, D+\epsilon), (D+\epsilon, D+2\epsilon), \cdots, (1/D-\epsilon, 1/D)$. Since the points of S are everywhere dense in the part of the continuum (perfect set) now under consideration, there must be some subportion (some perfect subset) in which the points of some one of the sets S_i are everywhere dense. As, moreover, the value of ϵ may be made as small as desired, we obtain the following simple

EXTENSION OF THE FUNDAMENTAL PRINCIPLE: *If to every point of a continuum (perfect set of points) a positive number δ_i is assigned in any manner, then for a sufficiently small value of D there is some portion of the continuum (some perfect subset) containing a set of points everywhere dense, for which the values of δ_i not only are greater than D but differ from one another by less than an arbitrarily assigned positive quantity ϵ.*

* The points q_i are said to be dense with respect to the subset, if within a sphere of any radius, described about any point of the subset as center, there lies always a point q_i.

† Cf. *Bericht*, p. 58.

In the case of a continuum of two or more dimensions it will be desirable not merely to know the existence of some portion of the continuum in which the points q_i are everywhere dense, but to ascertain also their distribution on a field of curves or surfaces which completely fill that portion. Inasmuch as every surface can be regarded as composed of curves, it will suffice here to consider the distribution of the q_i with respect to a continuous curve-system. For simplicity we shall confine our attention to a system of parallel straight lines, the results obtained being easily capable of extension to other curve-systems.

Let first a portion of space be selected from the continuum in the form of a right cylinder having its axis parallel to the given system of parallel straight lines. For the case of a two dimensional continuum the cylinder shall be understood to be a rectangle. Consider then the segment intercepted upon any one of the parallels by the cylinder. Since each segment is itself a continuum, there must be for a sufficiently small value of D some portion of the segment in which the points q_i having $\delta_i > D$ are everywhere dense. Denote for any segment the length of such a portion by l_i and the corresponding value of D by D_i. We will then assign to the segment (or, if we prefer, to its intersection with the base of the cylinder) the smaller of the two numbers l_i and D_i. Call it s_i. Now the set of segments (intersections) is perfect, and therefore by our "fundamental principle" the segments (intersections) for which $s_i > S$ must be everywhere dense in some portion of the cylinder (base), provided that S is a sufficiently small number. Take any such portion and let this dense set of segments be denoted by G. Divide any element of this new portion into equal parts not exceeding $S/2$ in length, and through the points of division pass planes parallel to the base of the cylinder, thereby dividing the portion under consideration into a finite number of component cylinders. Each segment of G contains at least one interval of length $s_i > S$ along which the points q_i having $\delta_i > s_i > S$ are everywhere dense. Consequently some one of the component cylinders must intercept an interval of length $S/2$, upon which the points q_i are everywhere dense. Since also the lines of G were everywhere dense before the subdivision into component cylinders took place, these intercepted intervals of length $S/2$ must be everywhere dense in some portion of one or more of the component cylinders. We obtain thus the following addendum to our fundamental principle:

ADDENDUM. *When the continuum is a portion of a plane or a space of higher dimensions, it is possible to select the region in which the points q_i are everywhere dense so as to fulfill the following condition: For a sufficiently small but fixed value of D there is throughout the region a dense set of parallels, upon each of which the q_i are everywhere dense; and the direction of the parallels may be assigned arbitrarily before seeking the region.*

§ 2. *Application to derivatives.*

Let $f(x)$ be a function of a real variable having a derivative at every point of an interval (a, b) of the X-axis. If ϵ be a small arbitrarily assigned positive quantity, any point p_i in (a, b) can be made the center of an interval $2\delta_i$ within which

$$\left|\frac{f(p_i+h)-f(p_i)}{h}-f'(p_i)\right|<\epsilon \qquad (|h|<\delta_i).$$

I shall assign to each point p_i such a positive number δ_i. Then, by § 1, for a sufficiently small D the points q_i for which $\delta_i > D$ must be dense in some portion of (a, b). Let (a', b') be such a portion, and denote by Q_i the dense set of points q_i within (a', b'). Then for any two points q_1 and q_2 whose distance $|q_1 - q_2|$ does not exceed D we have

$$\left|\frac{f(q_1)-f(q_2)}{q_1-q_2}-f'(q_1)\right|<\epsilon, \qquad \left|\frac{f(q_1)-f(q_2)}{q_1-q_2}-f'(q_2)\right|<\epsilon,$$

and therefore

$$|f'(q_1)-f'(q_2)|<2\epsilon. \tag{1}$$

Consequently the variation of $f'(x)$, taken for the set Q_i and for any subinterval of (a', b') not greater than D in length, will not be greater than 4ϵ.

The fundamental principle has been used thus to obtain a portion of the continuum in which there is a dense set of points with a certain specified property. Our second step in this and subsequent demonstrations consists in the extension of the property to all points of the portion. In the special case before us now, every point p of (a', b'), and in particular a point p which does not belong to Q_i, has a small neighborhood $2h$ such that

$$\left|\frac{f(q_i)-f(p)}{q_i-p}-f'(p)\right|<\epsilon' \qquad (|q_i-p|<h),$$

where ϵ' is an arbitrary small prescribed quantity. But also when $|q_i - p| \leqq D$, we have

$$\left|\frac{f(q_i)-f(p)}{q_i-p}-f'(q_i)\right|<\epsilon.$$

Consequently

$$|f'(p)-f'(q_i)|<\epsilon+\epsilon'. \tag{2}$$

As ϵ' is arbitrarily small, we conclude from (1) and (2) that the variation of $f'(x)$ in any portion of (a', b') not exceeding D in length will not exceed $\zeta_1 = 6\epsilon$.

This brings us to the third and final step of this and similar demonstrations; namely, the establishment of the pointwise discontinuous character of the function under consideration after it has been demonstrated that there is some por-

tion of the original field in which the variation of the function does not exceed an arbitrarily prescribed quantity ζ_1. The reasoning is so familiar that I give it here once for all, remarking merely that it is applicable to fields of two or more dimensions as well as to intervals. Take any series of decreasing positive numbers $\zeta_1, \zeta_2, \cdots$ having 0 as their limit. Inside the part of the field in which the variation does not exceed ζ_1 there must be, by like reasoning, a part in which the variation does not exceed ζ_2; in the interior of this another part in which the variation does not exceed ζ_3, and so on. The limit of this series of intervals or regions, taken each entirely in the interior of the preceding, is a point (or set of points) at which the function is continuous. Furthermore, as the argument applies to any portion of the initial field, the points of continuity must be everywhere dense, and the function is at most only pointwise discontinuous.

Applied to $f'(x)$ in the particular case before us, this reasoning gives at once the following theorem:

THEOREM I. *If a function of a real variable x has a derivative at every point of an interval (a, b), this derivative is at most only pointwise discontinuous in the interval* (BAIRE).*

It is well known that the points of continuity in the case of a pointwise discontinuous function have the power of a continuum. The same remark applies to all other sets of points obtained in this paper (cf. theorem VII) by the above method. In fact, to each value of ζ_i there was found between a and b, or between any two points of (a, b), a certain interval with specified property. Consequently the totality of points which cannot be included in the interior of such intervals is nowhere dense in (a, b). Call this set of points B_i. Corresponding to $\zeta_1, \zeta_2, \cdots$ we have an enumerable set of such sets B_i, each nowhere dense in (a, b), and these together constitute what BAIRE calls a point-set of the *first category*.† The complementary point-set, which consists of such points as we obtain by our method, has the power of a continuum. ‡

Theorem I may be extended to one-side derivatives. The extension is not without some interest, for it is known that the four derivatives of a function, when distinct throughout an interval, must be totally discontinuous. § The possibilities in respect to continuity are accordingly settled for the various species of derivatives. For extension of the above theorem to one-side derivatives BAIRE's proof requires no modification. The proof given here may be modified as follows.

Suppose that only a right-hand derivative $f'_+(x)$ exists in (a, b). Assign

* BAIRE deduces the theorem as a corollary of the fundamental theorem which will be quoted at the beginning of § 3; loc. cit., p. 64 or *Bericht*, p. 224.

† Loc. cit., p. 65.

‡ Cf. *Bericht*, p. 108.

§ Cf. DINI'S *Grundlagen für die Theorie der Functionen einer reellen Grösse*, p. 249.

to each point p_i of the interval a positive number δ_i such that

$$\left|\frac{f(p_i+h)-f(p_i)}{h}-f'_+(p_i)\right|<\frac{\epsilon}{4} \qquad (0<h<\delta_i). \tag{3}$$

For a sufficiently small D there will be some subinterval (a', b') in which the points q_i for which $\delta_i > D$ are everywhere dense. Take within it two such points, q_1 and $q_2 = q_1 + h$, for which $0 < h \leqq D/2$. We have the three inequalities

$$\left|\frac{f(q_1+h)-f(q_1)}{h}-f'_+(q_1)\right|<\frac{\epsilon}{4}, \tag{4}$$

$$\left|\frac{f(q_1+2h)-f(q_1+h)}{h}-f'_+(q_1+h)\right|<\frac{\epsilon}{4}, \tag{5}$$

$$\left|\frac{f(q_1+2h)-f(q_1)}{2h}-f'_+(q_1)\right|<\frac{\epsilon}{4}, \tag{6}$$

which combined together give

$$|f'_+(q_1+h)+f'_+(q_1)-2f'_+(q_1)|<\epsilon$$

or

$$|f'_+(q_2)-f'_+(q_1)|<\epsilon. \tag{7}$$

Consequently for Q_i—, i. e., the set of points q_i within (a', b')—the variation of $f'_+(x)$ will not exceed 2ϵ in any segment not exceeding $D/2$ in length.

Take next any point p of (a', b') not belonging to Q_i and let $\delta < D$ be a number assigned to it in accordance with (3). Since p is the limit of a set of points q_i which lie to the right of it, we may choose from Q_i some point $q_i = p + h$ for which $0 < h < \delta/2$. Then if we replace q_1 by p and q_2 by q_i, the inequalities (4)–(7) will hold as before. Consequently

$$|f'_+(q_i)-f'_+(p)|<\epsilon.$$

It follows therefore that the variation of $f'_+(x)$ for any portion of (a', b') which does not exceed $D/2$ in length will not be greater than $\zeta_1 = 4\epsilon$. The conclusion that $f'_+(x)$ is at most only pointwise discontinuous then follows in the manner previously described.

We shall consider next a function $f(x, y)$ which has a partial derivative $\partial f/\partial x$ at every point of a field T defined by

$$a \leqq x \leqq b, \qquad c \leqq y \leqq d.$$

By virtue of this hypothesis $f(x, y)$ must be continuous with respect to x. We suppose also that it is continuous with respect to y upon a set of parallels $x = x_i$ to the Y-axis which are everywhere dense in T, and proceed then to examine the character of $\partial f/\partial x$ in regard to continuity when both x and y are

varied. Any point $p=(x, y)$ may be made the center of an interval parallel to the axis of x and having such a length* 2δ that

$$\left|\frac{f(x+h, y)-f(x, y)}{h}-\frac{\partial f(x, y)}{\partial x}\right|<\epsilon \qquad (|h|<\delta). \tag{8}$$

By the "addendum" of § 1 it is possible for a sufficiently small value of D to find a part of T

$$T': \qquad a' \leqq x \leqq b', \qquad c' \leqq y \leqq d'$$

where the points q_i having $\delta_i > D$ will be everywhere dense upon each line of some system of parallels, $y=y_j$, which is itself everywhere dense throughout T'. Let $x=x_1$ and $x=x_1+h$ be any two lines within T' along which $f(x, y)$ is continuous with respect to y. Then, *h being fixed in value*, the difference quotient

$$r(x_1, y)=\frac{f(x_1+h, y)-f(x_1, y)}{h} \tag{9}$$

will be a continuous function of y on the line $x=x_1$. Denote by LL' any segment of the line in which the variation of (9) is less than a prescribed ϵ, and upon LL' as a side construct within T' a rectangle $LL'M'M$ having its second side $L'M'<D$.

Consider the variation of $\partial f(x, y)/\partial x$ in this rectangle, and first of all upon the dense set of parallels $y=y_j$. If the intersection (x_1, y_j) of x_1 with one of these parallels is not itself one of the points q_i, it is the limit of a set of such points upon $y=y_j$, and we may take a point $q_j=(\bar{x}, y_j)$ so close to (x_1, y_j) that the difference of the values of the different quotient $r(x, y)$ at these two points will be numerically less than an arbitrarily assigned quantity ϵ'. At the second point q_j we have

$$\left|r(\bar{x}, y_j)-\frac{\partial f(\bar{x}, y_j)}{\partial x}\right|<\epsilon \qquad (h \leqq D).$$

Since, furthermore, on the one hand, the variation of $r(x, y)$ upon LL' is less than ϵ, and, on the other hand, the variation of $\partial f/\partial x$ upon $y=y_j$ is less than $\zeta=6\epsilon$ by the proof of theorem I, it follows at once that the total variation of $\partial f/\partial x$, taken for the set of parallels $y=y_j$, will be less than $\epsilon+2(6\epsilon)+2(\epsilon+\epsilon')$. Since also ϵ' is arbitrary, it will not exceed 15ϵ.

Bring next into consideration the points of the rectangle which do not lie upon the dense set of parallels $y=y_j$. For any such point (x', y') we can find another point (x'', y') so near to it that

$$\left|\frac{f(x'', y')-f(x', y')}{x''-x'}-\frac{\partial f(x', y')}{\partial x}\right|<\epsilon'. \tag{10}$$

* In this and other cases BAIRE considers the *maximum* interval 2δ fulfilling the condition imposed and proves that δ as a function of (x, y) is "*semi-continue supérieurement.*" In the method which is here being followed, it is unnecessary to introduce the maximum δ.

Now if neither point lies upon a line of continuity for $f(x, y)$ with respect to y, we can take a pair of such points, $(\bar{x}, y')$ and $(\bar{x}'', y')$, so close to them that

$$\left| \frac{f(\bar{x}'', y') - f(\bar{x}, y')}{\bar{x}'' - \bar{x}} - \frac{f(x'', y') - f(x', y')}{x'' - x'} \right| < \epsilon', \tag{11}$$

while if either point lies upon such a line of continuity we have merely to identify it in (11) with the corresponding point of the second pair. Having now two points upon lines of y-continuity, by varying y we can find one of the dense set of parallels $y = y_j$ so near to $y = y'$ that

$$\left| \frac{f(\bar{x}'', y_j) - f(\bar{x}', y_j)}{\bar{x}'' - \bar{x}'} - \frac{f(\bar{x}'', y') - f(\bar{x}', y')}{\bar{x}'' - \bar{x}'} \right| < \epsilon'. \tag{12}$$

But upon this parallel it is possible to select from the dense set of points q_i such a point $q_i = (x_i, y_j)$ that

$$\left| \frac{f(x_i, y_j) - f(\bar{x}', y_j)}{x_i - \bar{x}'} - \frac{f(\bar{x}'', y_j) - f(\bar{x}', y_j)}{\bar{x}'' - \bar{x}'} \right| < \epsilon' \quad (|x_i - \bar{x}'| < D). \tag{13}$$

Since, lastly, for q_i

$$\left| \frac{\partial f(x_i, y_j)}{\partial x_i} - \frac{f(x_i, y_j) - f(\bar{x}', y_j)}{x_i - \bar{x}'} \right| < \epsilon, \tag{14}$$

it follows immediately from (10)–(14) that

$$\left| \frac{\partial f(x_i, y_j)}{\partial x} - \frac{\partial f(x', y')}{\partial x} \right| < \epsilon + 4\epsilon',$$

where ϵ' is arbitrarily small. As the variation of $\partial f/\partial x$ upon the dense set of parallels $y = y_j$ does not exceed 15ϵ, we conclude from the last inequality that the variation of $\partial f/\partial x$ in the rectangle $LL'M'N$ will not exceed $15\epsilon + 2\epsilon = 17\epsilon$.

This rectangle can now be substituted in place of the original field. By indefinitely decreasing $\zeta_1 = 17\epsilon$ we obtain as the limit of a series of rectangles, lying each within the preceding, a point of continuity of $\partial f/\partial x$ considered as a function of (x, y). This proves the following theorem;

THEOREM II. *If $f(x, y)$ at every point of T possesses a partial derivative $\partial f(x, y)/\partial x$ and if, furthermore, there is in T an everywhere dense set of parallels $x = x_i$ upon which $f(x, y)$ is continuous in y, then $\partial f(x, y)/\partial x$ as a function of (x, y) is at most only pointwise discontinuous.*

For the sake of completeness I call attention to BAIRE's important remark* that when the derivative $\partial f/\partial y$ exists as well as $\partial f/\partial x$, there must be at least a set of points everywhere dense throughout T at which both derivatives are continuous, and at which, therefore, $f(x, y)$ has a differential. The proof follows immediately by making the values of $\zeta_1, \zeta_2, \zeta_3, \cdots$ refer alternately to the variation of the two derivatives.

*Loc. cit., pp. 108–110.

§ 3. *Theorems on the continuity of* $f(x, y)$.

BAIRE in his thesis has given the following important theorem:

If $f(x, y)$ *is continuous with respect to* y *at every point of a rectangle*

$$T: \quad a \leqq x \leqq b, \qquad c \leqq y \leqq d,$$

and with respect to x *on a set of parallels to the* X*-axis everywhere dense in* T*, then* $f(x, y)$ *as a function of the two variables together is at most only point-wise discontinuous; and the points of continuity in* T *will be everywhere dense upon any curve having an equation of the form* $y = \phi(x)$*, wherein* $\phi(x)$ *is a continuous function of* x.*

SCHÖNFLIES,† commenting on this theorem, has remarked that although BAIRE's hypothesis of continuity with respect to x is stated for a dense set of parallel straight lines, this is clearly an unessential specialization, for in place of this set of parallels "*eine überall dichte Curvenschar*" could be substituted. There is, however, nothing in BAIRE's work to indicate the mode of distribution of the points of continuity along such a system of curves $y = \phi_i(x)$ as SCHÖNFLIES suggests. I shall complete the result by establishing the following theorem:

THEOREM III. *If* $f(x, y)$ *is continuous with respect to* y *at every point of* T *and with respect to* x *upon a set of curves everywhere dense and having equations of the form* $y = \phi_i(x)$*, in which* $\phi_i(x)$ *is continuous in* x*, then* $f(x, y)$ *as a function of* (x, y) *is continuous at all the intersections of these curves with some set of parallels to the* Y*-axis which are likewise everywhere dense in* T.

Proof. To every line $x = x_i$ in the field T let a positive number δ_i be so assigned that the variation of $f(x, y)$ in any segment not exceeding δ_i in length will not exceed a fixed but arbitrarily prescribed ϵ. According to our fundamental principle, for a sufficiently small value of D the set of lines $x = \xi_i$ for which $\delta_i > D$ must be dense in some portion of T. Suppose such a portion T' to be included between $x = \xi_1$ and $x = \xi_2$, and denote the dense set of lines between these limits by Ξ. We shall establish first that the characteristic property of the lines of Ξ holds within T' for every parallel to the Y-axis.

Consider for this purpose any line $x = x'$ which is not known to belong to Ξ. It is cut by the curves $y = \phi_i(x)$ in a dense set of points (x', y_i). Take now any two points whatsoever of the line, (x', y') and (x', y''), for which $|y' - y''| \leqq D$. If these points are not themselves included in the dense set of intersections (x', y_i), we can take two of the intersections $(x', y_i), (x', y_j)$ which lie so near to these two points that simultaneously

$$|y_i - y_j| \leqq D,$$

$$|[f(x', y') - f(x', y'')] - [f(x', y_i) - f(x', y_j)]| < \epsilon,$$

* Loc. cit., p. 27.

† *Bericht*, p. 140. The specification as to their distribution upon $y = \phi(x)$ is omitted in SCHÖNFLIES' summary of the theorem (p. 142).

in which ϵ' is arbitrarily small. Then since (x', y_i), (x', y_j) are points of continuity of $f(x, y)$ as a function of x, they may be made the centers of intervals parallel to the X-axis and so small that in either interval the variation of $f(x, y)$ will be less than ϵ'. Furthermore, since the lines of Ξ are everywhere dense in T', the two intervals will be cut by certain lines of the set. Let (ξ', y_i), (ξ', y_j) be the intersections of the two intervals with some one of these lines. We now have the inequalities

$$|[(f(x', y_i) - f(x', y_j)] - [f(\xi', y_i) - f(\xi', y_j)]| < 2\epsilon',$$

$$|f(\xi', y_i) - f(\xi', y_j)| \leqq \epsilon.$$

It follows then by combination of these with the inequality above that

$$|f(x', y') - f(x', y'')| < \epsilon + 3\epsilon',$$

and hence, since ϵ' is arbitrary,

$$|f(x', y') - f(x', y'')| \leqq \epsilon.$$

We have thus shown that the variation of $f(x, y)$ in T' upon any parallel to the Y-axis will not exceed ϵ for any segment of the parallel not greater than D in length.

Any point (x_1, y_1) within T' at which $f(x, y)$ is continuous with respect to x can now be made the center of a rectangle, with sides parallel to the axes of x and of y, in which the variation of $f(x, y)$ does not exceed 3ϵ. For suppose it first to be made the center of an interval $(x_1 - \delta', x_1 + \delta')$ parallel to the X-axis in which the variation does not exceed ϵ. Then the rectangle bounded by

$$x = x_1 - \delta', \qquad x = x_1 + \delta', \qquad y = y_1 - D, \qquad y = y_1 + D,$$

or so much of it as lies in T', fulfills the conditions stated.

Let the value of ϵ be now diminished. Corresponding to an indefinitely diminishing series $\epsilon', \epsilon'', \epsilon''', \cdots$ we will have a series of rectangles $T', T'', T''', \cdots$ having the same altitude as T and lying each within the preceding. The limit of this series of rectangles is a line $y = \eta$ (or a continuous set of such lines) which has the following property: Every point of the line at which $f(x, y)$ is continuous with respect to x is also a point of continuity of $f(x, y)$ with respect to (x, y), inasmuch as it can be enclosed in a series of rectangles in which the variation of $f(x, y)$ is successively not greater than $3\epsilon', 3\epsilon'', 3\epsilon''', \cdots$. In particular, every point in which the line is cut by the curves $y = \phi_i(x)$ is a point of continuity of $f(x, y)$.

If the original field T is divided into two parts by a parallel to the Y-axis, our reasoning applies to each part. The lines $y = \eta$ are therefore everywhere dense in T. This proves the theorem.

One particular case of theorem III is worthy of special note, that in which $f(x, y)$ is continuous in x and y separately at every point of T. Applied to this case the theorem becomes the following corollary:

THEOREM IV. *If $f(x, y)$ is continuous in x and y separately, there exists in T an everywhere dense set of parallels to the Y-axis (also to the X-axis), at every point of which $f(x, y)$ is continuous in (x, y).*

This is merely another statement of the following conclusion of BAIRE:*

THEOREM IV'. *If $f(x, y)$ is continuous in x and y separately within T, the projections upon the axis of x of all points of discontinuity whose measure is greater than an arbitrarily prescribed positive quantity σ will be nowhere dense.*

To establish the equivalence of the two statements, it suffices to note that, by the proof of IV, between any two parallels to the Y-axis containing points of discontinuity which have a measure greater than σ, there may be inserted one of our strips T' for which $\sigma_1 \equiv 3\epsilon$ may be taken less than σ. Because 3ϵ is an upper limit for the measure of discontinuity at all interior points of T', it follows that the projections of the discontinuities having a measure $> \sigma$ can be nowhere dense. Conversely, if IV' be assumed, theorem IV at once results by application of familiar principles.

In the method which I have given here, attention has been concentrated upon the parallels of continuity for $f(x, y)$ rather than upon those containing points of discontinuity. This makes it easy to make the following advance beyond BAIRE's result:

THEOREM V. *Upon the hypothesis enunciated in theorem* IV *there exists also a set of lines everywhere dense in T and parallel to the Y-axis, upon each of which $f(x, y)$ as a function of (x, y) is uniformly continuous.*†

To prove this theorem consider again the strip T'. Divide any interior line $x = x'$ into segments equal to or less than $D/2$ in length by means of the points

$$(x', y_1),\ (x', y_2),\ \cdots,\ (x', y_n).$$

Each point (x', y_i) can be made the center of an interval $2\sigma_i$ parallel to the axis, in which the variation of $f(x, y)$ does not exceed a prescribed ϵ. Let σ be the smallest of the numbers σ_i and mark out a strip of T' bounded by the parallels

$$x = x' - \sigma, \qquad x = x' + \sigma.$$

Take now any two points (x, y), (x'', y'') in the interior of the strip such that $|y - y''| \leqq D/2$, and make each point the center of an interval of length D

* The proof given above is much simpler than that of BAIRE (loc. cit., pp. 88–94), who does not make use of the theorem of which III was an extension.

† The term "uniformly continuous" here signifies that if any ε is arbitrarily prescribed, a value δ can be found such that for the vicinity of all points (x', y') of the line considered the variation of $f(x, y)$ will be less than ε when $|x - x'| \leqq \delta$, $|y - y'| \leqq \delta$, provided, of course, (x, y) lies within the field T.

parallel to the axis of y. It has been seen that the variation of $f(x, y)$ in either of these intervals does not exceed ϵ. Since also they will cut in common some one of the n intervals σ_i, we have

$$|f(x, y) - f(x'', y'')| \leqq 3\epsilon.$$

If now we confine ourselves to the strip S bounded by

$$x = x' - \frac{\sigma}{2}, \qquad x = x' + \frac{\sigma}{2},$$

every point in S may be made the center of a square, the sides of which are parallel to the axes and have a length equal to the smaller of the two numbers σ and D. Within the square (or such part of the square as is contained in T) the variation of $f(x, y)$ will not exceed 6ϵ.

The strip S may now be substituted in place of T, and the reasoning be repeated. To a sequence of indefinitely diminishing values $\epsilon_1, \epsilon_2, \epsilon_3, \cdots$ there corresponds a series of strips of decreasing breadth, contained each in the preceding, and the limit is a line $x = \bar{x}$ on which $f(x, y)$ is uniformly continuous. Obviously, also, such lines will be found between any two parallels to the axis of y, and they are accordingly everywhere dense in T.

§ 4. *Theorems concerning integrals.*

Suppose that

$$F(x) \equiv \int_c^d f(x, y)\,dy \tag{15}$$

is the integral of any function $f(x, y)$ which in the rectangle T is continuous in x and y separately. We have just found in T a strip S for which the following property holds: If $|x' - x''|$ does not exceed σ, then

$$|f(x', y) - f(x'', y)| \leqq 3\epsilon.$$

Consequently we have for the difference of the values of (15) along the parallel paths $x = x'$, $x = x''$

$$|F(x') - F(x'')| = \left| \int_c^d f(x', y)\,dy - \int_c^d f(x'', y)\,dy \right| \leqq 3\epsilon(d - c).$$

In other words, in the interval (a, b) may be found a subinterval in which the variation of $F(x)$ does not exceed $\zeta_1 = 6\epsilon(d - c)$. The conclusion concerning the pointwise discontinuous character of $F(x)$ which has been stated previously in theorem VI of the introduction, follows at once in the manner explained in § 1.

Consider next the difference quotient

$$\frac{F(x + \Delta x) - F(x)}{\Delta x} = \int_c^d \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}\,dy.$$

If $f(x, y)$ has a derivative with respect to x, the application of the mean-value theorem gives for the value of the right-hand member

$$\int_c^d f'_x(x + \theta \cdot \Delta x, y)\, dy,$$

where θ is a number between 0 and 1 which varies with y. If, now, $f'_x(x, y)$ is continuous in x and y separately, there is a strip of T between two appropriately chosen parallels $x = x' \pm \delta$, within which the variation of $f'_x(x, y)$ for constant y will not exceed 3ϵ. Hence when x and $x + \Delta x$ both lie between these limits for x, we have

$$\left| \int_c^d f'_x(x + \theta \cdot \Delta x, y)\, dy - \int_c^d f'_x(x, y)\, dy \right| \leqq 3\epsilon(d - c).$$

Consequently

$$\frac{F(x + \Delta x) - F(x)}{\Delta x} = \int_c^d f'_x(x, y)\, dy + \epsilon_1,$$

where

$$|\epsilon_1| \leqq 3\epsilon(d - c).$$

The interval $(x' - \delta, x' + \delta)$ and the corresponding strip of T may now be substituted in place of (a, b) and the field T, and the argument may be repeated with a smaller value of ϵ. Continuing in this way we obtain with diminishing ϵ a series of intervals contained each in the interior of the preceding and having for their limit a point x at which

$$\lim_{\Delta x = 0} \frac{F(x + \Delta x) - F(x)}{\Delta x} = F'(x) = \int_c^d f'_x(x, y)\, dy.$$

As the argument applies to any portion of (a, b), the first part of theorem VII is established.

The second part of the theorem follows from the remark that in the succession of strips diminishing with ϵ the variation of $f'_x(x, y)$ for constant y does not exceed 3ϵ, so that the variation of

$$F_1(x) = \int_c^d f'_x(x, y)\, dx$$

does not exceed $3\epsilon(d - c)$. The limiting point x is therefore a point of continuity of $F_1(x)$.

§ 5. *On series.*

Let

$$S(x) = u_1(x) + u_2(x) + \cdots$$

be a series, whose terms are continuous in (a, b) and which converges at every point of this interval. Denote by $S_n(x)$ the sum of the first n terms of the

series. To each point x_i of the interval a number n_i can be so assigned that

$$|S_{n_i+m}(x_i) - S(x_i)| < \epsilon \qquad (m = 1, 2, 3, \cdots).$$

If ϵ is fixed and a sufficiently large N is selected, the set of points $x = q_i$ for which $n_i < N$ will be everywhere dense in some subinterval, (a', b'). Let Q denote this set of points. Contract this interval so much as may be necessary to make the variation of the continuous function $S_N(x)$ within (a', b') less than ϵ. Then for q_1 and q_2, any two points of Q, we have

$$|S_N(q_1) - S(q_1)| < \epsilon,$$

$$|S_N(q_2) - S(q_2)| < \epsilon,$$

and

$$|S_N(q_1) - S_N(q_2)| < \epsilon.$$

Hence

$$|S(q_1) - S(q_2)| < 3\epsilon; \tag{16}$$

that is to say, the variation of $S(x)$ in (a', b') over the set Q does not exceed 6ϵ.

Consider next a point p in (a', b') which does not belong to Q. A fixed value $N' \geqq N$ can be found such that

$$|S_{N'}(p) - S(p)| < \epsilon'.$$

As the points q_i are everywhere dense in $(a'b')$, one of these points be found so near to p that

$$|S_{N'}(p) - S_{N'}(q_i)| < \epsilon'.$$

Since also

$$|S_{N'}(q_i) - S(q_i)| < \epsilon,$$

it follows from the last three inequalities that

$$|S(p) - S(q_i)| < \epsilon + 2\epsilon'. \tag{17}$$

But ϵ' is arbitrary. We conclude therefore from (16) and (17) that the variation of $S(x)$ within (a', b') can not exceed $\zeta_1 = 8\epsilon$. By § 2 this gives at once the following theorem:

THEOREM VIII. *If the series*

$$S(x) = u_1(x) + u_2(x) + u_3(x) + \cdots$$

converges at every point of an interval (a, b) and each term of the series is a continuous function of x, the sum of the series can be at most only pointwise discontinuous in (a, b) (BAIRE).*

This theorem was first proved by OSGOOD† for the case in which the limit is a

* Loc. cit., p. 62, or *Bericht*, p. 224.

† American Journal of Mathematics, vol. 19 (1897).

continuous function, and he showed, moreover, that the points of uniform convergence* are everywhere dense. Recently HOBSON† has proved that the points of uniform convergence are also dense in the more comprehensive case considered in theorem VIII. This conclusion results also from the proof given above, for it can be shown at once that all the points of continuity for $S(x)$ obtained through the demonstration are points of uniform convergence. Take, in fact, any one of these points. It can be made first the center of an interval (α, β) in which the variation of $S(x)$ is less than ϵ. But it also lies in an interval (a', b') in which the points q_i are everywhere dense. For any of these points we had

$$|S_n(q_i) - S(q_i)| < \epsilon \qquad (n \geqq N). \tag{18}$$

In the vicinity of any point p of (a', b') we can find a point q_i so near that, for given n and ϵ',

$$|S_n(p) - S_n(q_i)| < \epsilon'.$$

But for two points q_i and p interior to (α, β),

$$|S(q_i) - S(p)| < \epsilon.$$

Hence from the last three inequalities

$$|S_n(p) - S(p)| \leqq 2\epsilon \qquad (n > N). \tag{19}$$

The two inequalities (18), (19) hold simultaneously for the common portion of (a', b') and (α, β), and consequently the center of (α, β) is, as stated, a point of uniform convergence.

* By a point of uniform convergence is to be understood one which, for any arbitrarily assigned ε, can be enclosed in an interval in which

$$|S(x) - S_n(x)| < \varepsilon \qquad (n > N)$$

for all values of x and for a sufficiently large N.

† Proceedings of the London Mathematical Society, vol. 34 (1902), p. 245, or Acta Mathematica, vol. 27 (1903), p. 209. An earlier proof was given in TOWNSEND's dissertation (Göttingen, 1900), p. 68.

INVARIANTS OF BINARY FORMS UNDER MODULAR TRANSFORMATIONS*

BY

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For a non-modular field F and a binary form f with coefficients in F, the problem of the determination of functions of the coefficients and variables of f, invariant under all binary linear transformations in F, is formally identical with the corresponding problem of the ordinary algebraic invariant theory. But for a finite modular field the problem is essentially different; the terms of an invariant need not be of the same degree nor of constant weight; the annihilators are quite complicated, involving higher partial derivatives. Fortunately, the difficulty in the direct computation of the invariants is in marked contrast with the regularity observed in the actual form of the invariants and with the simplicity of the relations between the invariants, common to the algebraic and modular theories, and the additional invariants peculiar to the modular theory.

In the study of the invariants of a given quantic in the Galois field of order p^n, we have a doubly infinite system of problems, corresponding to a single problem in the algebraic theory. Interest naturally centers in a comparative study, rather than in the individual problems. The aim of the present paper is to give in correlation the results of a rather extensive comparative study. Formal proofs of the laws observed are given only in a few instances. It is hoped that proofs of the remaining properties observed will become more practicable when a satisfactory symbolic treatment is constructed.

Since there is evidently only a finite number of linearly independent invariants I of a given binary form of degree m in the $GF[p^n]$, each set of numbers m, n, p uniquely defines a commutative linear associative algebra, whose units are these invariants I.

In the final sections of the paper, application is made to the invariantive reduction of binary quantics to canonical forms.†

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† The invariants of k-ary quadratic forms are treated in papers to appear in the Proceedings of the London Mathematical Society and the American Journal of Mathematics.

1. THEOREM. *The binary form* with coefficients in the $GF[p^n]$,*

$$f=\sum_{i=0}^{m} a_i x^{m-i} y^i, \tag{1}$$

has the absolute invariant

$$I=\prod_{i=0}^{m}(a_i^{\mu}-1), \qquad \mu\equiv p^n-1. \tag{2}$$

Here and below we employ the generators (t, λ any marks $\neq 0$)

$$x=x'+ty', \qquad y=y'; \tag{3}$$

$$x=x', \qquad y=\lambda y'; \tag{4}$$

$$x=y', \qquad y=-x'; \tag{5}$$

of the group of all binary linear transformations with coefficients in the $GF[p^n]$. Under (3) f becomes a form f' with the coefficients

$$a_0'=a_0, \quad a_1'=a_1+mta_0, \quad \cdots, \quad a_i'=a_i+\sum_{j=0}^{i-1} C_{m-j,\,i-j}\, t^{i-j} a_j, \quad \cdots. \tag{6}$$

Under transformations (4) and (5) we have, respectively,

$$a_i'=\lambda^i a_i \qquad (i=0,1,\cdots,m), \tag{7}$$

$$a_i'=(-1)^{m-i} a_{m-i} \qquad (i=0,1,\cdots,m). \tag{8}$$

Under the replacements (7) and (8) the function (2) is unaltered, since $\lambda^{\mu}=1$ for every mark $\lambda\neq 0$. Consider next the replacement (6). The first factor $a_0^{\mu}-1$ is unaltered. Then, since $(a_0^{\mu}-1)a_0=0$,

$$(a_0'^{\mu}-1)(a_1'^{\mu}-1)=(a_0^{\mu}-1)(a_1^{\mu}-1+Aa_0)=(a_0^{\mu}-1)(a_1^{\mu}-1).$$

The proof may be completed by induction from $i-1$ to i. Let

$$P_i=\prod_{j=0}^{i}(a_j^{\mu}-1).$$

Then

$$P_i'=P_{i-1}(a_i'^{\mu}-1)=P_{i-1}(a_i^{\mu}-1+B_0a_0+B_1a_1+\cdots+B_{i-1}a_{i-1})$$
$$=P_{i-1}(a_i^{\mu}-1)=P_i.$$

2. The absolute invariant (2) has a simple interpretation. According as $a\neq 0$ or $a=0$, we have $a^{\mu}-1=0$ or -1. *Hence the form f vanishes identically if and only if $I\neq 0$.*

* We do not at present introduce binomial coefficients C_{mi}, some of which are divisible by p for certain values of m and p.

Frequently a property expressible by a single invariant in the modular theory requires in the algebraic theory the use of a covariant or else an invariantive simultaneous system of equations. A less obvious instance relates to the minors of the determinant of a k-ary quadratic form.

3. THEOREM. *For m a power of p, the binary form* (1) *of degree m with coefficients in the $GF[p^n]$ has the absolute invariant*

$$P = \prod_{i=1}^{m-1} (a_i^{\mu} - 1), \qquad \mu \equiv p^n - 1. \tag{9}$$

The proof is similar to that in § 1. We now have

$$C_{mi} \equiv 0 \pmod{p} \qquad (i = 1, \cdots, m-1),$$

so that there is no term a_0 in a_i' $(i = 1, \cdots, m-1)$ in formulæ (6).

4. An obvious invariant of f is the eliminant E of

$$f = 0, \qquad x^{p^n} = x, \qquad y^{p^n} = y.$$

For example, let $m = 3$, $p^n = 3$. Multiply f by xy, x^2y^2 and x^2 in turn and reduce by means of $x^3 = x$, $y^3 = y$. The corresponding equations

$$a_0 x + a_1 x^2 y + a_2 x y^2 + a_3 y = 0, \qquad (a_0 + a_2) x^2 y + (a_1 + a_3) x y^2 = 0,$$

$$(a_1 + a_3) x^2 y + (a_0 + a_2) x y^2 = 0, \qquad a_0 x + (a_1 + a_3) x^2 y + a_2 x y^2 = 0,$$

have the eliminant, identical with (17) of § 9,

$$a_0 a_3 \{ (a_1 + a_3)^2 - (a_0 + a_2)^2 \}.$$

Evidently, $E = a_m R$, where R is the resultant of

$$f_1 \equiv a_0 + a_1 y + \cdots + a_m y^m = 0, \qquad y^{p^n} = y.$$

By the elimination of $1, y, y^2, \cdots, y^{p^n-1}$ by SYLVESTER's dialytic method, we obtain for R a determinant of order p^n in which each element in the main diagonal is a_0. The numerical factor is thereby determined so that $R = a_0^{p^n} + \cdots$.

Except in the simplest cases, it is more convenient to have E expressed as the product of the linear functions obtained from f by giving to x, y the $p^n + 1$ sets of values $0, 1$; and $1, \rho$, where ρ ranges over the marks of the field:

$$E = a_m \Pi (a_0 + a_1 \rho + \cdots + a_m \rho^m), \quad \rho \text{ ranging over the } GF[p^n].$$

For the case of quadratic forms ($m = 2$) in the $GF[p^n]$, simple expressions may be given to E. For $p = 2$, $E = a_1^2 \chi$, where χ is the absolute

invariant given by (20). For $p>2$,

$$E=\tfrac{1}{2}(\Delta^{\frac{1}{2}(p^n+1)}-\Delta), \qquad \Delta=a_1^2-4a_0a_2.$$

Note, in verification of this result, that $f_1=0$ has a root in the field if, and only if, the discriminant Δ is zero or a square, so that $E=0$ if, and only if,

$$\Delta(\Delta^{\frac{1}{2}(p^n-1)}-1)=0.$$

For cubic forms, consider first the case $p^n=3^2$.

We define the $GF[9]$ by the primitive irreducible congruence $i^2\equiv i+1 \pmod 3$; then the marks $\neq 0$ are 1, i, i^2, i^3, and their negatives. Hence $E=a_0a_3\pi$, where

$$\pi=\Pi\{(a_0+a_2\rho^2)^2-(a_1\rho+a_3\rho^3)^2\} \qquad (\rho=1, i, i^2, i^3)$$
$$=\Pi(a_0^2-\alpha\rho^2+\beta\rho^4-a_3^2\rho^6) \quad (\alpha=a_1^2+a_0a_2,\ \beta=a_2^2+a_1a_3).$$

Forming the product of the factors given by $\rho=1$, i^2, and those given by $\rho=i$, i^3, we get, since $i^4=-1$,

$$\pi=\{(a_0^2+\beta)^2-(a_3^2+\alpha)^2\}\{(a_0^2-\beta)^2+(a_3^2-\alpha)^2\}$$
$$=(a_0^4+a_3^2\alpha+\beta^2)^2-(a_3^4+a_0^2\beta+\alpha^2)^2.$$

Since $a_0a_3(a_0^8-a_3^8)=0$ in the field, are readily obtain the result:

$$E=a_0^3a_1a_3^6-a_0a_1^4a_3^5+(a_0^2a_1a_2^3-a_0a_1^3a_2^2)a_3^4-(a_0^2a_2^5+a_0a_1^2a_2^4+a_0^6a_2)a_3^3$$
$$+(a_0a_1a_2^6-a_0^4a_1^3a_2+a_0^3a_1^5)a_3^2+(a_0a_2^8+a_0^5a_2^4+a_0^4a_1^2a_2^3+a_0^3a_1^4a_2^2-a_0^2a_1^6a_2-a_0a_1^8)a_3. \tag{10}$$

For $p\neq 3$, we shall employ the cubic form (40) with binomial coefficients. Then π is the product of the functions $a_0+3a_1\rho+3a_2\rho^2+a_3\rho^3$, ρ ranging over the marks $\neq 0$. For $p^n=2$, 2^2, 2^3, the expressions for π and E are given in §23. For $p^n=5$,

$$\pi=(a_0^2-a_2^2-a_1a_3)^2-(a_1^2-a_3^2+a_0a_2)^2$$
$$=a_0^4-a_1^4+a_2^4-a_3^4+2a_0^2a_2^2-2a_1^2a_3^2+2a_0a_2a_3^2-2a_0^2a_1a_3+2a_1a_2^2a_3-2a_0a_1^2a_2,$$

so that E is given by (90). For $p^n=7$, it is convenient to set

$$\alpha=a_0^2-a_3^2, \qquad \beta=a_1a_3+2a_2^2, \qquad \gamma=-a_0a_2-2a_1^2.$$

Forming the product of the factors by twos, viz., those with $\rho=\pm 1$, those with $\rho=\pm 2$, and those with $\rho=\pm 3$, we obtain the interesting result:

$$\pi\equiv(\alpha+\beta+\gamma)(\alpha+2\beta+4\gamma)(\alpha+4\beta+2\gamma)\equiv\alpha^3+\beta^3+\gamma^3-3\alpha\beta\gamma \pmod 7.$$

The resulting expression for E is given by (93).

5. THEOREM. *The weights* $\sigma = e_1 + 2e_2 + \cdots + me_m$ *of the various terms*

$$a_0^{e_0} a_1^{e_1} \cdots a_m^{e_m}$$

of an invariant ϕ *differ by multiples of* $\mu = p^n - 1$.

Here and in § 6, let ϕ become $\phi' = D^d \phi$ under every binary transformation of determinant D. Consider the special transformation (4) in which λ is a primitive root of the $GF[p^n]$. In view of (7), the above general term of ϕ is multiplied by λ^σ. Hence

$$\lambda^\sigma = \lambda^d, \qquad \sigma \equiv d \pmod{\mu}.$$

6. THEOREM. *The degrees* ρ *of the various terms of an invariant differ by multiples of* μ/δ, *where* δ *is the greatest common divisor of* m *and* $\mu = p^n - 1$. *Further,* $m\rho - 2\sigma$ *is a multiple of* μ.

For the transformation $x = \lambda x'$, $y = \lambda y'$, we have $a_i' = \lambda^m a_i$. Hence a term of degree ρ of ϕ is multiplied by $\lambda^{m\rho}$, so that $m\rho \equiv 2d \pmod{\mu}$.

7. The differential operators which annihilate an invariant are here more complicated than in the algebraic theory. This is due primarily to the fact that, in a series of powers of an arbitrary mark t of the $GF[p^n]$, certain terms now combine, viz., t^i, $t^{i+\mu}$, $t^{i+2\mu}$, $\cdots$, where $\mu = p^n - 1$. Since $a^{p^n} = a$ for every mark a, we may assume that the exponent of each a_i in an invariant does not exceed μ. Then, for the case $n = 1$, we can employ TAYLOR's theorem for the expansion of a polynomial of degree $\mu = p - 1$,

$$\psi(a + t) = \psi(a) + t\psi'(a) + \cdots + \frac{1}{i!} t^i \psi^{(i)}(a) + \cdots + \frac{1}{\mu!} t^\mu \psi^{(\mu)}(a),$$

since the denominator $i!$ is prime to the modulus p. The case $n > 1$ apparently (cf. § 10) offers a theoretical difficulty; it actually presents a difficulty from another source (§ 12). We therefore begin with examples illustrating the simpler case $n = 1$.

8. For the first illustrative example, consider the form

$$a_0 x^2 + a_1 xy + a_2 y^2, \tag{11}$$

in which the coefficients are integers taken modulo 3. Under the transformation (3), a_1 and a_2 receive, in view of (6), the respective increments

$$\alpha_1 = 2ta_0, \qquad \alpha_2 = ta_1 + t^2 a_0.$$

Let ϕ be a polynomial in a_0, a_1, a_2, with exponents $\leqq 2$. By TAYLOR's theorem,

$$\phi' - \phi = \alpha_1\phi_{a_1} + \alpha_2\phi_{a_2} + \tfrac{1}{2}\alpha_1^2\phi_{a_1^2} + \alpha_1\alpha_2\phi_{a_1a_2} + \tfrac{1}{2}\alpha_2^2\phi_{a_2^2}$$
$$+ \tfrac{1}{2}\alpha_1^2\alpha_2\phi_{a_1^2a_2} + \tfrac{1}{2}\alpha_1\alpha_2^2\phi_{a_1a_2^2} + \tfrac{1}{4}\alpha_1^2\alpha_2^2\phi_{a_1^2a_2^2},$$

where $\phi_{a_1^2a_2}$ denotes $\partial^3\phi/\partial a_1^2\partial a_2$, etc. Then $\phi' - \phi = t\delta\phi + t^2\delta_1\phi$, where

$$\delta\phi \equiv 2a_0\phi_{a_1} + a_1\phi_{a_2} + 2a_0^2\phi_{a_1a_2} + a_0a_1\phi_{a_2^2} + 2a_0^2a_1\phi_{a_1^2a_2} + (a_0 + a_0a_1^2)\phi_{a_1a_2^2} + 2a_0a_1\phi_{a_1^2a_2^2},$$

$$\delta_1\phi \equiv a_0\phi_{a_2} + 2a_0^2\phi_{a_1^2} + 2a_0a_1\phi_{a_1a_2} - (a_0^2 + a_1^2)\phi_{a_2^2} + 2a_0\phi_{a_1^2a_2} + 2a_0^2a_1\phi_{a_1a_2^2}$$
$$+ (a_0^2 + a_0^2a_1^2)\phi_{a_1^2a_2^2}.$$

A necessary condition for $\phi' = \phi$ is $\delta\phi = 0$. This condition is also sufficient. Indeed, if $[\delta\phi]$ denotes $\delta\phi$ in reduced form, i. e., with every exponent ≤ 2 (in virtue of $a^3 \equiv a$), then $\delta[\delta\phi] \equiv \delta_1\phi$. To verify the latter, note that if ψ is a reduced function of a, viz., $\psi = ra^2 + sa + t$, then $[a\psi] \equiv ra + sa^2 + ta$, and $[a\psi]_a \equiv a\psi_a + \psi - \psi_{aa} \pmod 3$. Again, if $\chi = sa + t$, $[a^2\chi] \equiv sa + ta^2$, and

$$[a^2\chi]_a \equiv a^2\chi_a + (a^2)_a\chi + \chi_a \pmod 3.$$

Proceeding to the computation of the invariants, we may set

$$\phi = \sum_{i,j}^{0,1,2} A_{ij}a_1^ia_2^j \qquad (A_{ij} \text{ quadratic functions of } a_0).$$

Employing the annihilator δ and giving to $\delta\phi$ its reduced form $[\delta\phi]$ with exponents ≤ 2, we require that $[\delta\phi] \equiv 0$, identically in a_1, a_2. Among the resulting conditions occur

$$A_{11} \equiv 0, \qquad A_{12} \equiv 0, \qquad A_{10}a_0 \equiv 0, \qquad A_{22}a_0 \equiv 0 \pmod 3.$$

In view of these, the remaining conditions reduce to

$$A_{21}a_0 - A_{22} - A_{02} \equiv 0, \qquad A_{20}a_0 + A_{21} + A_{01} - A_{21}a_0^2 - A_{02}a_0 \equiv 0.$$

From $A_{22}a_0 \equiv 0$ we get $A_{22} \equiv r(a_0^2 - 1)$, where r is a constant. The term $ra_0^2a_1^2a_2^2$ is unaltered by transformation (4), so that $r = 0$ unless ϕ is an absolute invariant. In the latter case we replace ϕ by $\phi - rI$, where I is the invariant given by (2). In either case, it remains to consider ϕ with $A_{22} \equiv 0$. Under transformation (5),

$$(8') \qquad a_0' = a_2, \qquad a_1' = -a_1, \qquad a_2' = a_0.$$

This replacement must leave ϕ unaltered. But by $A_{22} \equiv 0$, $a_1^2a_2^2$ is a factor of no term of ϕ. Hence $a_1^2a_0^2$ is a factor of no term of ϕ. We may therefore set

$$A_{21} = \alpha + \beta a_0, \qquad A_{20} = \lambda + \mu a_0 \quad (\alpha, \beta, \lambda, \mu \text{ constants}).$$

Similarly, by $A_{12} \equiv 0$, $a_1 a_0^2$ is a factor of no term, so that $A_{10} \equiv 0$. The above long conditions now give

$$A_{02} = \alpha a_0 + \beta a_0^2, \qquad A_{01} = -\alpha + (\beta - \lambda) a_0 - (\alpha + \mu) a_0^2.$$

The conditions that ϕ shall be unaltered by (8′) are $\mu = \alpha$, $A_{00} = k - \alpha a_0$, k a constant. Hence

$$\phi = \beta a_0^2 a_2^2 + (\beta - \lambda) a_0 a_2 + \lambda a_1^2 + \beta a_0 a_2 a_1^2 + \alpha Q + k,$$

where

$$Q = a_0 a_2^2 + a_0^2 a_2 + a_1^2 a_2 + a_1^2 a_0 - a_2 - a_0. \tag{12}$$

The coefficient of λ is the discriminant $\Delta = a_1^2 - a_0 a_2$, that of β is congruent to $\Delta^2 - \Delta$. *As the linearly independent invariants of the quadratic form* (11) *modulo* 3, *we may take* I, Q, Δ, Δ^2. *Now* $I \equiv Q^2 + \Delta^2 - 1$. *As the independent invariants we may take* Q *and* Δ.

9. Consider the binary cubic $(1)_{m=3}$ with integral coefficients modulo 3. The discriminant Δ and (9) give the absolute invariants

$$\Delta = a_1^2 a_2^2 - a_0 a_2 - a_1 a_3, \qquad P = (a_1^2 - 1)(a_2^2 - 1). \tag{13}$$

Hence there is an absolute invariant of the second degree:

$$K \equiv \Delta - P + 1 = a_1^2 + a_2^2 - a_0 a_2 - a_1 a_3. \tag{14}$$

In view of (6), a_0 and a_1 are unaltered by transformation (3), while a_2 and a_3 have the respective increments $2ta_1$ and $t(a_0 + a_2) + t^2 a_1$. The coefficient of t in $\phi' - \phi$ gives the annihilator

$$\begin{aligned} &2a_1\phi_{a_2} + (a_0 + a_2)\phi_{a_3} - a_1^2\phi_{a_2 a_3} + a_1(a_0 + a_2)\phi_{a_3^2} - a_1^2(a_0 + a_2)\phi_{a_2^2 a_3} \\ &\quad + \{a_1 + a_1(a_0 + a_2)^2\}\phi_{a_2 a_3^2} - a_1(a_0 + a_2)\phi_{a_2^2 a_3^2}. \end{aligned} \tag{15}$$

By §§ 5, 6, the terms of an invariant ϕ are all of even degrees, while all are of even or all of odd weights. Further, by (5) and (8), ϕ must be unaltered by the substitution

$$a_0' = -a_3, \qquad a_3' = a_0, \qquad a_1' = a_2, \qquad a_2' = -a_1. \tag{16}$$

Let first the weights be even, so that by § 5, $e_1 \equiv e_3 \pmod 2$ in every term of ϕ. Then, in view of (16), ϕ must have the form

$$\begin{aligned} &b(a_0^2 + a_3^2) + c(a_1^2 + a_2^2) + d(a_0 a_2 + a_1 a_3) + e a_0^2 a_3^2 + f a_1^2 a_2^2 + g(a_0^2 a_1^2 + a_2^2 a_3^2) \\ &\quad + h(a_0^2 a_2^2 + a_1^2 a_3^2) + j(a_0^2 a_1 a_3 + a_3^2 a_0 a_2) + k(a_1^2 a_0 a_2 + a_2^2 a_1 a_3) + l a_0 a_1 a_2 a_3 \\ &\quad + m(a_0^2 a_1^2 a_2^2 + a_1^2 a_2^2 a_3^2) + n(a_0^2 a_1^2 a_3^2 + a_0^2 a_2^2 a_3^2) + q(a_0^2 a_2^2 a_1 a_3 + a_1^2 a_3^2 a_0 a_2) \\ &\quad + r a_0^2 a_1^2 a_2^2 a_3^2. \end{aligned}$$

By subtracting from ϕ suitable multiples of P, K, Δ^2, and I, given by (2), we may set $c=d=k=r=0$. Then (15) vanishes if and only if

$$n=f=0, \qquad g=q=e=h=-m, \qquad b=j=l=m.$$

The coefficient of m is the absolute invariant

$$T=a_0^2+a_3^2-a_0^2a_3^2-a_0^2a_1^2-a_2^2a_3^2-a_0^2a_2^2-a_1^2a_3^2+a_0^2a_1a_3+a_0a_2a_3^2 \\ +a_0a_1a_2a_3+a_0^2a_1^2a_2^2+a_1^2a_2^2a_3^2-a_0^2a_1a_2^2a_3-a_0a_1^2a_2a_3^2.$$

Finally, when the weights are odd, the possible terms are

$$B(a_0a_1-a_2a_3)+G(a_0^2a_1a_2-a_3^2a_1a_2)+H(a_0^2a_2a_3-a_3^2a_0a_1) \\ +J(a_1^2a_0a_3-a_2^2a_0a_3)+K(a_1^2a_2a_3-a_2^2a_0a_1)+L(a_0^2a_1^2a_2a_3-a_3^2a_2^2a_0a_1).$$

The conditions that (15) shall vanish are $B=G=K=L=0$, $J=H$. Hence

$$E=a_1^2a_0a_3-a_2^2a_0a_3+a_0^2a_2a_3-a_3^2a_0a_1 \tag{17}$$

is the only invariant of odd weights. By §§ 4, 27, the cubic form is irreducible modulo 3 if, and only if, $E\neq 0$.

The relations between T and the remaining invariants are

$$IT=0, \qquad PT=P-I, \qquad \Delta T=I-P+T, \qquad ET=E, \qquad T^2=T.$$

We obtain the simpler relations * (19) by introducing $W\equiv I-P+T$, viz.,

$$W=a_3^2(a_0^2a_1^2a_2^2-a_0^2a_2^2-a_0a_1^2a_2+a_0a_2-a_0^2a_1^2)+a_3(-a_0^2a_1a_2^2+a_0a_1a_2+a_0^2a_1). \tag{18}$$

THEOREM. *As a complete set of linearly independent invariants of the cubic form modulo* 3, *we may take* I, P, Δ, Δ^2, E, W. *The product of any two invariants can be reduced to a linear function of these six by means of*

$$\begin{cases} I^2=I, \quad IP=I, \quad I\Delta=IE=IW=0, \quad P^2=P, \quad P\Delta=PE=PW=0, \\ \Delta^3=\Delta, \quad \Delta E=E, \quad \Delta W=W, \quad E^2=W, \quad EW=E, \quad W^2=W. \end{cases} \tag{19}$$

10. The next illustrative examples relate to the invariants of a quantic in the $GF[p^n]$, $n>1$. We require the expansion of $\psi(a+t)$ in powers of t, where $\psi(a)$ is a polynomial of degree at most $\mu=p^n-1$. This can be done by TAYLOR'S theorem, the coefficient of t^i being the quotient obtained algebraically by dividing $\psi^{(i)}(a)$, all of whose coefficients are exact multiples of $i!$, by the number $i!$. Similar remarks apply to TAYLOR'S theorem for two or more variables.

* Conforming with the relations (§ 21) between the invariants of the cubic form in the $GF[9]$. Note that $I-P+T$ and $-\Delta-\Delta^2$ are the only invariants W satisfying the relations (19) involving W. Hence there is a single invariant W which can replace T.

11. Consider the quadratic form (11) with coefficients in the $GF[2^n]$. Under the transformation (3), a_0 and a_1 are unaltered while the increment to a_2 is $ta_1 + t^2 a_0$. Let ϕ be a polynomial in a_0, a_1, a_2, with exponents $\leqq 2^n - 1$.

Let first $n = 2$. Then the coefficient of t in $\phi' - \phi$ equals

$$a_1 \phi_{a_2} + a_0^2 [\tfrac{1}{2} \phi_{a_2^2}] + a_0 a_1^2 [\tfrac{1}{6} \phi_{a_2^3}],$$

where the indicated divisions of the derivatives by 2 and 6 are to be performed algebraically and the quotients only are interpreted in the $GF[4]$. If we set

$$\phi = \sum_{i=0}^{3} B_i a_2^i \qquad (B\text{'s functions of } a_0, a_1),$$

the above expression for the coefficient of t becomes, in the $GF[4]$,

$$a_1(B_1 + B_3 a_2^2) + a_0^2(B_2 + B_3 a_2) + a_0 a_1^2 B_3.$$

This must vanish identically in a_2. From $a_1 B_3 = a_0 B_3 = 0$ we get

$$B_3 = r(a_0^3 - 1)(a_1^3 - 1) \qquad (r = \text{constant}).$$

The remaining condition * becomes $a_1 B_1 + a_0^2 B_2 = 0$. Replacing ϕ by $\phi - rI$, where I is the absolute invariant (2), we may set $B_3 = 0$. By (8), ϕ must be symmetrical in a_0 and a_2. Hence no term of ϕ contains the factor a_2^3 or the factor a_0^3. By §§ 5, 6, the degrees ρ (and likewise the weights σ) of the various terms of ϕ differ by multiples of 3, and $\rho \equiv \sigma \pmod 3$.

Let first $\rho \equiv \sigma \equiv 0$. Then ϕ involves only the terms a_1^3, $a_0 a_1 a_2$, $a_0^2 a_1^2 a_2^2$, so that

$$B_0 = l a_1^3, \qquad B_1 = m a_0 a_1, \qquad B_2 = k a_0^2 a_1^2.$$

Then $m = k$ by $a_1 B_1 + a_0^2 B_2 = 0$. The linearly independent invariants † are

$$I, \qquad a_1^3, \qquad J = a_0 a_1 a_2 + a_0^2 a_1^2 a_2^2.$$

For $\rho \equiv \sigma \equiv 1$, ϕ involves only the terms a_1, $a_0 a_1^2 a_2$, $a_0^2 a_2^2$, $a_0^2 a_1^3 a_2^2$, so that

$$B_0 = b a_1, \qquad B_1 = c a_0 a_1^2, \qquad B_2 = d a_0^2 + e a_0^2 a_1^3.$$

By $a_1 B_1 + a_0^2 B_2 = 0$, we get $d = 0$, $e = c$. The invariants † are a_1 and $a_1 J$.

The case $\rho \equiv \sigma \equiv 2$ may be reduced to the preceding by squaring, an operation here reversible. Hence the invariants are a_1^2 and $a_1^2 J$.

* The corresponding conditions from the coefficient of t^2 are $a_1 B_3 = a_0 B_3 = 0$, $a_0 B_1 + a_1^2 B_2 = 0$. Thus the vanishing of the coefficient of t does not imply the vanishing of that of t^2 (but does that of t^3, viz., $a_0^3 B_3 + a_1^3 B_3$). In this respect the case of invariants in the $GF[p^n]$, $n > 1$, appears to be in contrast to the case $n = 1$.

† The remaining conditions from t^2 (preceding foot-note) are satisfied.

For general n, we can prove (cf. §§ 4, 26) the following

THEOREM. *The independent invariants of the quadratic form* (11) *in the* $GF[2^n]$ *may be taken to be* a_1 *(which is multiplied by the determinant of the transformation), I and the absolute invariant*

$$(20) \qquad \chi(a_0 a_2 a_1^{2^n-3}), \qquad \textit{where} \qquad \chi(c) \equiv c + c^2 + c^4 + \cdots + c^{2^{n-1}},$$

the last invariant being replaced by $a_0 a_2 a_1$ *in the case* $n = 1$.

12. The illustrative examples employed thus far were chosen on account of their comparative simplicity. As the order of the $GF[p^n]$ increases, the complexity of the computation increases very rapidly. Certain remarks will be found to be very useful. While, for $n > 1$, certain divisions are called for in the definition of an annihilator δ (see §§ 10, 11), the actual performance of the divisions can be dispensed with in the computation of $\delta\phi$. For example, in the $GF[3^2]$ the result of operating on a term $a_0^e a_1^i a_2^j$ by $1/3!\ 1/2!\ \partial^5/\partial a_1^3 \partial a_2^2$ is zero unless $i \geqq 3$, $j \geqq 2$, while in the latter case the result is $C_{i3} C_{j2} a_0^e a_1^{i-3} a_2^{j-2}$. It thus suffices to have a table* of the residues modulo 3 of the binomial coefficients C_{rs} $(r, s \leqq 8)$. In view of this remark, the following permanent notations will be used for the terms of $\delta\phi$, the notations not exhibiting the numerical divisors. We shall set

$$(21) \qquad (1^i) = \frac{1}{i!} \frac{\partial^i \phi}{\partial a_1^i}, \qquad (1^i 2^j) = \frac{1}{i!\ j!} \frac{\partial^{i+j} \phi}{\partial a_1^i \partial a_2^j}, \quad \text{etc.}$$

In the algebraic theory ϕ is invariant under the special transformation (3) if, and only if, $\delta\phi = 0$, where $\delta\phi$ denotes the coefficient of t in $\phi' - \phi$. The same theorem appears to hold in the modular theory when $n = 1$. In a few cases (cf. § 8), I have made a direct proof of this statement; in very many cases I have secured indirect verification in showing that the computed functions are actually invariants. But for $n > 1$ such a theorem does not hold (cf. foot-notes to § 11). However, it appears to be true that the vanishing of the coefficients of t, t^p, t^{p^2}, $\cdots$, $t^{p^{n-1}}$ implies the vanishing of the coefficients of the remaining powers (the last one being $p^n - 1$). The importance of this conjecture as a guide in actual computations is obvious; its effect on the explicit form of the invariant is noted in § 17.

13. Consider the quadratic form (11) in the $GF[3^2]$. Under transformation (3), a_1 and a_2 receive the respective increments $\delta_1 = -ta_0$, $\delta_2 = ta_1 + t^2 a_0$. Expanding the powers $\delta_2^2, \cdots, \delta_2^8$, reducing the coefficients modulo 3 and the exponents by means of $x^9 = x$, we write down by inspection (in view of the

* The residues of multinomial coefficients modulo p, a prime, are obtained rapidly by means of two general theorems given in the writer's dissertation, Annals of Mathematics, ser. 1, vol. 11 (1896–7), pp. 75, 76.

simple form of δ_1) the coefficient (22) of t and the coefficient (23) of t^3 in $\phi' - \phi$, employing the notations (21).

$$
\begin{aligned}
&- a_0(1) + a_1(2) - a_0^8(1^72) + a_0^8a_1(1^82) - a_0^7(1^52^2) - a_0^7a_1(1^62^2)\\
&- a_0^7a_1^2(1^72^2) - a_0^6(1^32^3) + a_0^6a_1^3(1^62^3) - a_0^5(12^4) + a_0^5a_1(1^22^4)\\
&+ a_0^5a_1^3(1^42^4) - a_0^5a_1^4(1^52^4) - a_0^4a_1(2^5) - a_0^4a_1^2(12^5) + a_0^4a_1^3(1^22^5)\\
&+ a_0^4a_1^4(1^32^5) + a_0^4a_1^5(1^42^5) - a_0^4(1^72^5) - a_0^4a_1(1^82^5) - a_0^3a_1^3(2^6)\\
&- a_0^3a_1^6(1^32^6) - a_0^3(1^52^6) - a_0^3a_1^3(1^82^6) - a_0^2a_1^6(12^7) + a_0^2a_1^7(1^22^7)\\
&- a_0^2(1^32^7) + a_0^2a_1(1^42^7) - a_0^2a_1^3(1^62^7) + a_0^2a_1^4(1^72^7) - a_0a_1^7(2^8)\\
&- (a_0 + a_0a_1^8)(12^8) - a_0a_1(1^22^8) - a_0a_1^2(1^32^8) - a_0a_1^3(1^42^8)\\
&- a_0a_1^4(1^52^8) - a_0a_1^5(1^62^8) - a_0a_1^6(1^72^8) - a_0a_1^7(1^82^8).
\end{aligned}
\tag{22}
$$

$$
\begin{aligned}
&- a_0^3(1^3) - a_0^2(12) + a_0^2a_1(1^22) - a_0a_1(2^2) - a_0a_1^2(12^2) - a_0(1^72^2)\\
&- a_0a_1(1^82^2) + a_1^3(2^3) - a_0^8(1^52^3) + a_0^8a_1^3(1^82^3) - a_0^7(1^32^4)\\
&+ a_0^7a_1(1^42^4) + a_0^7a_1^3(1^62^4) - a_0^7a_1^4(1^72^4) - a_0^6(12^5) - a_0^6a_1(1^22^5)\\
&- a_0^6a_1^2(1^32^5) + a_0^6a_1^3(1^42^5) + a_0^6a_1^4(1^52^5) + a_0^6a_1^5(1^62^5) - a_0^5a_1^3(1^22^6)\\
&- a_0^5a_1^6(1^52^6) - a_0^5(1^72^6) - a_0^4a_1^3(2^7) + a_0^4a_1^4(12^7) - a_0^4a_1^6(1^32^7)\\
&+ a_0^4a_1^7(1^42^7) - a_0^4(1^52^7) + a_0^4a_1(1^62^7) - a_0^4a_1^3(1^82^7) - a_0^3a_1^5(2^8)\\
&- a_0^3a_1^6(12^8) - a_0^3a_1^7(1^22^8) - (a_0^3 + a_0^3a_1^8)(1^32^8) - a_0^3a_1(1^42^8)\\
&a_0^3a_1^2(1^52^8) - a_0^3a_1^3(1^62^8) - a_0^3a_1^4(1^72^8) - a_0^3a_1^5(1^82^8).
\end{aligned}
\tag{23}
$$

We may set

$$
\phi = \sum_{i,j}^{0,\cdots,8} A_{ij}a_1^ia_2^j \qquad (A\text{'s functions of } a_0). \tag{24}
$$

Since a_2^8 occurs only in the first term of (22) and (23), we get

$$
A_{i8}a_0 = 0 \qquad (i = 1, \cdots, 8).
$$

Employing these to simplify the coefficient of a_2^7 in the first four terms of (22) and that in the first three terms of (23), we get

$$
- a_0\sum iA_{i7}a_1^{i-1} - a_1\sum A_{i8}a_1^i, \qquad - a_0^3\sum C_{i3}A_{i7}a_1^{i-3}.
$$

These must vanish identically in $a_1 = a_1^9$. Hence

$$
A_{i7}a_0 = 0 \ (i = 1, 3, \cdots, 8), \quad A_{i8} = 0 \ (i = 1, \cdots, 7), \quad A_{27}a_0 - A_{08} - A_{88} = 0.
$$

Before examining the further conditions we shall introduce a decided simplification. From $A_{88}a_0 = 0$, $A_{88} = k(a_0^8 - 1)$, k a constant. But the term

$ka_0^8a_1^8a_2^8$ of ϕ is unaltered by transformation (4). Hence k is zero unless ϕ is an absolute invariant. In the latter case, we replace ϕ by $\phi - kI$, where I is the absolute invariant (2). In either case we have $A_{88}=0$. *Then* $A_{ij}a_0=0$ $(i\neq 0)$ *implies* $A_{ij}=0$. For, then $A_{ij}=c(a_0^8-1)$, while $a_0^8a_1^ia_2^j$ is not a term of ϕ. If it were, then would $a_0^ja_1^ia_2^8$ occur in ϕ since ϕ must be unaltered by (5) and hence by (8′) of § 8, whereas $A_{i8}=0$ $(i=1,\cdots,8)$.

We next examine the coefficients of a_2^6 in (22) and (23), then those of a_2^5, etc. At each step we utilize the conditions previously found. In view of the above result and the fact that the binomial coefficients C_{i2} $(i=3,4,6,7)$, C_{64}, C_{65}, C_{75} are multiples of the modulus 3, the determination of the coefficient of a_2^k may be done readily by inspection. The resulting conditions,* including the earlier ones, are:

$$\begin{aligned}
&A_{i8}=0\,(i=1,\cdots,8),\ A_{i7}=0(i=1,3,\cdots,8),\ A_{i6}=0(i=1,3,5,6,7,8),\\
&A_{i5}=0\,(i=1,2,3,4,5,7,8),\ A_{i4}=0\,(i=1,3,4,5,7),\\
&A_{i3}=0\,(i=1,3,5,7),\ A_{i2}=0\,(i=1,2,3,5,7,8),\\
&A_{i1}=0\,(i=1,3,5,7),\ A_{i0}=0\,(i=1,3,5,7);
\end{aligned}\tag{25}$$

$$\begin{aligned}
&A_{08}=A_{27}a_0,\ A_{27}=A_{46}a_0,\ A_{07}=-A_{26}a_0,\ A_{65}=A_{84}a_0,\ A_{27}=A_{84}a_0^3,\\
&A_{05}=A_{24}a_0,\ A_{07}=A_{61}a_0^3,\ A_{64}=-A_{83}a_0,\ A_{24}=A_{43}a_0,\ A_{26}=A_{83}a_0^3,\\
&A_{06}=A_{63}a_0^3,\ A_{46}=A_{84}a_0^2,\ A_{05}=-A_{62}a_0^3,\ A_{24}=-A_{81}a_0^3,\ A_{64}=A_{41}a_0^3,\\
&A_{42}=A_{81}a_0^6,\ A_{62}=A_{81}a_0,\ A_{02}=A_{21}a_0,\ A_{23}a_0+A_{01}+A_{81}+A_{84}a_0^8=0,\\
&A_{61}a_0^3=A_{23}a_0,\ A_{41}=-A_{83}a_0^6,\ A_{21}=A_{40}a_0-A_{63}a_0^6,\ A_{61}+A_{80}a_0+A_{27}a_0^2=0,\\
&A_{20}a_0+A_{01}+A_{81}+A_{81}a_0^8=0,\ A_{63}=A_{40}a_0^3-A_{21}a_0^2,\ A_{43}=-A_{81}a_0^2,\\
&A_{23}+A_{80}a_0^3+A_{84}a_0^7,\ A_{03}+A_{83}+A_{60}a_0^3-A_{07}a_0^4=0.
\end{aligned}\tag{26}$$

We note that each A_{ij} not in the set (25) has a non-vanishing value in at least one invariant given below.

In the notations of § 5, we have here

$$e_1+2e_2\equiv 2e_0+e_1\equiv d \pmod 8.\tag{27}$$

Consider first the absolute invariants, so that $d\equiv 0$. For $e_1=8$, $e_2=0$ or 4, we have $e_0=0$, 4 or 8. The value $e_0=8$ is excluded, since (as shown above) $a_0^8a_1^8a_2^{e_2}$ is not a term of ϕ. Hence we may set

$$A_{84}=e+fa_0^4,\qquad A_{80}=l+ma_0^4,\qquad A_{00}=\lambda+\mu a_0^4+\nu a_0^8.$$

* Those in set (25) and likewise the remaining ones occur approximately in the order of their determination. Obvious simplifications have been made in the latter conditions. Certain conditions have been omitted as being simple consequences of those retained.

By (27) the only further non-vanishing A_{ij} are the following, whose values are derived at once from the non-identical conditions (26):

$$A_{65} = ea_0 + fa_0^5, \quad A_{27} = ea_0^3 + fa_0^7, \quad A_{08} = ea_0^4 + fa_0^8, \quad A_{46} = ea_0^2 + fa_0^6,$$

$$A_{42} = ea_0^6 + fa_0^2, \quad A_{23} = -(f+l)a_0^3 - (e+m)a_0^7, \quad A_{61} = -(f+l)a_0 - (e+m)a_0^5,$$

$$A_{04} = -e + (l-f)a_0^4 + ma_0^8.$$

The necessary and sufficient conditions that the resulting function (24) shall be unaltered by (5) and hence by (8′) are $m = e$, $\mu = -e$, $\nu = 0$. The independent parameters are thus e, f, l. The coefficients of l and f are Δ^4 and $\Delta^8 - \Delta^4$, respectively, where

$$\Delta = a_1^2 - a_0 a_2 \tag{28}$$

is the discriminant of the quadratic form. The coefficient of e is

$$\begin{aligned} Q = {} & a_1^8 a_2^4 + a_0^4 a_1^8 - a_0^4 + a_0 a_1^6 a_2^5 + a_0^3 a_1^2 a_2^7 + a_0^4 a_2^8 \\ & + a_0^2 a_1^4 a_2^6 + a_0^6 a_1^4 a_2^2 + a_0^7 a_1^2 a_2^3 + a_0^5 a_1^6 a_2 - a_2^4 + a_0^8 a_2^4. \end{aligned} \tag{29}$$

By way of check we note here the important relations

$$Q^2 + \Delta^8 - 1 = I, \qquad Q\Delta = 0. \tag{30}$$

Next, there is no invariant with $d \equiv 1 \pmod 8$. Indeed, each of the nine $A_{e_1 e_2}$ for which $e_1 + 2e_2 \equiv 1 \pmod 8$ is zero by (25). There is no invariant with $d \equiv 3$; for, if so, its cube would be an invariant with $d \equiv 1$. Similarly, the cases $d \equiv 7$, $d \equiv 5$ are excluded.

In an invariant with $d \equiv 2 \pmod 8$, the possible non-vanishing A_{ij} are $A_{20}, A_{01}, A_{81}, A_{62}, A_{43}, A_{24}, A_{05}$. Now A_{81} is a linear function of a_0 and a_0^5 by (27). But $a_0^5 a_1^8 a_2$ is not a term of ϕ in view of (8′). Hence $A_{81} = la_0$. Similarly, $A_{20} = r + sa_0^4$. Then the seven non-vanishing conditions (26) give

$$A_{62} = la_0^2, \quad A_{05} = -la_0^5, \quad A_{24} = -la_0^4, \quad A_{43} = -la_0^3, \quad A_{01} = (l-r)a_0 - sa_0^5.$$

The resulting function ϕ must be unaltered by (8′); hence $s = 0$. The coefficients of r and l are Δ and $\Delta^5 - \Delta$, respectively. By cubing, we conclude that the only invariants with $d \equiv 6 \pmod 8$ are the linear functions of Δ^3 and Δ^7.

Finally, for $d \equiv 4$, every A_{ij} vanishes except $A_{40}, A_{21}, A_{02}, A_{63}, A_{06}$, while $A_{40} = c$, $A_{21} = ka_0$. The four non-vanishing conditions (26) give

$$A_{02} = ka_0^2, \qquad A_{63} = (c-k)a_0^3, \qquad A_{06} = (c-k)a_0^6.$$

For $c = k = 1$, $\phi = \Delta^2$; for $c = 1$, $k = 0$, $\phi = \Delta^6$.

THEOREM. *Every invariant of a quadratic form in the* $GF[9]$ *is a linear function of* I, Q, $\Delta^i (i = 1, \cdots, 8)$. *As independent invariants we may take* Q *and* Δ.

14. The results of §§8, 13 on the invariants of a quadratic form in the $GF[3^n]$, $n = 1, 2$, may be given a simpler form, better adapted to the extension to the case of general n. We set

$$(31) \quad J = Q + \Delta^2 - 1 \text{ (for } n = 1), \qquad J = Q + \Delta^8 - 1 \text{ (for } n = 2).$$

Then in each case

$$(32) \qquad J\Delta = 0, \qquad J^2 - J = I,$$

as follows from (30) and the similar relations for $n = 1$. The importance of the introduction of the invariant J lies in the simplicity of relations (32) and in the factorizations

$$(33) \qquad J = (a_0 + 1)(a_2 + 1)(a_1^2 + a_0 a_2 - 1) \qquad \text{(for } n = 1),$$

$$(34) \quad J = (a_0^4+1)(a_2^4+1)(a_1^8+a_0^4a_2^4+a_0a_2a_1^6+a_0^3a_2^3a_1^2+a_0^2a_2^2a_1^4-1) \quad \text{(for } n=2).$$

For general n, the corresponding function is

$$(35) \qquad J = (a_0^\nu + 1)(a_2^\nu + 1)\left\{ \sum_{i=0}^{\nu} a_0^i a_2^i a_1^{2\nu-2i} - 1 \right\},$$

where $\nu = \frac{1}{2}(3^n - 1)$. The discussion in §15 will apply here since (28) and (35) are unaltered modulo 3 when a_1 is multiplied by 2.

15. Consider the quadratic form* in the $GF[p^n]$, $p > 2$,

$$(36) \qquad a_0 x^2 + 2a_1 xy + a_2 y^2.$$

We investigate the function J given by (35) for $\nu = \frac{1}{2}(p^n - 1)$. It is obviously unaltered by the substitutions (7) and (8), so that the proof of its absolute invariance† depends only upon the verification that it is unaltered by the transformation (3). Let J' denote the function (35) of the transformed coefficients a_0', a_1', a_2'. Here $a_0' = a_0$, $\Delta' = \Delta$. As shown below, $J\Delta = 0$, $J'\Delta' = 0$. Hence if $\Delta \neq 0$, $J = J' = 0$. Let next $\Delta = 0$, so that $a_1^2 = a_0 a_2$, $a_1'^2 = a_0' a_2'$. Then the sum in (35) equals $(\nu + 1) a_0^\nu a_2^\nu$. Now $\nu + 1 = \frac{1}{2}(p^n + 1)$ is the mark $\frac{1}{2}$ in the field. Hence the final factor of J equals $\frac{1}{2} a_0^\nu a_2^\nu - 1$. If a_0 is a not-square, $a_0^\nu = -1$, so that $J = 0$, $J' = 0$. If $a_0 = 0$, then $a_1 = 0$, while

* If we employ (11) instead of (36), we must introduce the factor 4^i under the summation sign in (35). For the essentially distinct case $p = 2$, here excluded, see §11.

† The existence of an invariant independent of I and Δ follows from the canonical form theory (§26).

the form a_2y^2 is unaltered by transformation (3). Finally, let a_0 be a square $\neq 0$, so that $a_0^\nu = 1$. Then

$$J = (a_2^\nu + 1)(a_2^\nu - 2), \qquad J' = (a_2'^\nu + 1)(a_2'^\nu - 2).$$

Since a_0a_2 is a square or zero, a_2 is a square or zero, so that $a_2^\nu = 1$ or 0; in either case $J = -2$. Likewise $J' = -2$. We have now proved the following

THEOREM. *The quadratic form* (36) *in the* $GF[p^n]$, $p < 2$, *has the absolute invariant* J *defined by* (35) *for* $\nu = \frac{1}{2}(p^n - 1)$.

We next prove the following relations, which reduce to (32) for $p = 3$:

$$J\Delta = 0, \qquad (J+1)^2 = I + 1, \tag{37}$$

so that, *for the functions obtained from* I *and* J *by deleting their constant terms* -1, *the one is the square of the other.* We here have

$$\Delta = a_1^2 - a_0a_2, \qquad I = (a_0^{2\nu} - 1)(a_1^{2\nu} - 1)(a_2^{2\nu} - 1). \tag{38}$$

Since $a^{2\nu+1} = a$, a simple change of summation indices gives

$$J\Delta = P(a_0a_2 - a_0^{\nu+1}a_2^{\nu+1}), \qquad P \equiv (a_0^\nu + 1)(a_2^\nu + 1).$$

Then $J\Delta = 0$ follows from

$$a_0^{\nu+k}a_2^{\nu+k}P = (a_0^{2\nu+k} + a_0^{\nu+k})(a_2^{2\nu+k} + a_2^{\nu+k}) = (a_0^k + a_0^{\nu+k})(a_2^k + a_2^{\nu+k}) = a_0^k a_2^k P \quad (k > 0). \tag{39}$$

To determine J^2, call σ the last factor, $\Sigma - 1$, in (35). Then

$$(a_2^\nu + 1)^2\sigma^2 = \{a_2^{2\nu} - 1 + 2(a_2^\nu + 1)\}\sigma^2 = -(a_2^{2\nu} - 1)(a_1^{2\nu} - 1) + 2(a_2^\nu + 1)\sigma^2,$$

since $(a^{2\nu} - 1)a = 0$. Modifying $(a_0^\nu + 1)^2$ similarly, we get

$$J^2 = -I - 2(a_0^\nu + 1)(a_2^{2\nu} - 1)(a_1^{2\nu} - 1) + 2(a_2^\nu + 1)\{-(a_0^{2\nu} - 1)(a_1^{2\nu} - 1) + 2(a_0^\nu + 1)\sigma^2\}$$
$$= -I + 2P\{(a_1^{2\nu} - 1)(2 - a_0^\nu - a_2^\nu) + 2\sigma^2\}.$$

Now

$$\sigma = a_1^{2\nu} - 1 + a_0^\nu a_2^\nu + \Sigma, \qquad \Sigma \equiv \sum_{i=1}^{\nu-1} a_0^i a_2^i a_1^{2\nu-2i},$$

$$\sigma^2 = -(a_1^{2\nu} - 1) + 2(a_1^{2\nu} - 1)a_0^\nu a_2^\nu + a_0^{2\nu}a_2^{2\nu} + 2a_0^\nu a_2^\nu \Sigma + \Sigma^2.$$

Since $2J = 2P\sigma$, we have

$$J^2 + 2J = -I + 2P\{(a_1^{2\nu} - 1)(1 - a_0^\nu - a_2^\nu + a_0^\nu a_2^\nu) + L\} = I + 2PL,$$

$$L \equiv a_0^\nu a_2^\nu + 2a_0^{2\nu}a_2^{2\nu} + 3a_0^\nu a_2^\nu (a_1^{2\nu} - 1) + 4a_0^\nu a_2^\nu \Sigma + \Sigma + 2\Sigma^2.$$

Hence (37_2) will follow if we show that $PL = 0$. In view of (39),

$$PL = P(3a_0^\nu a_2^\nu a_1^{2\nu} + 5\Sigma + 2\Sigma^2).$$

The second member may be shown, by means of (39), to equal

$$P(2\nu+1)(a_0^\nu a_2^\nu a_1^{2\nu}+\Sigma),$$

and hence vanishes in the $GF[p^n]$, since $2\nu+1=p^n$.

16. This section and the four succeeding sections will be devoted primarily to the exhibition of a remarkable absolute invariant K of degree $\mu=p^n-1$ of the binary cubic form in the $GF[p^n]$, and to the simple relation between K and the discriminant Δ of the form. Except for the case $p=3$ (cf. § 9), we shall write the form with binomial coefficients

$$a_0x^3+3a_1x^2y+3a_2xy^2+a_3y^3. \tag{40}$$

Under transformation (3), a_1, a_2, a_3 take the respective increments

$$\delta_1=ta_0, \qquad \delta_2=2ta_1+t^2a_0, \qquad \delta_3=3ta_2+3t^2a_1+t^3a_0. \tag{41}$$

For each term (§ 5) of an absolute invariant ϕ we have

$$3e_0+2e_1+e_2=r\mu, \qquad e_1+2e_2+3e_3=s\mu \qquad (r, s \text{ integers}). \tag{42}$$

The condition that the term shall be of degree μ is $r+s=3$. For $r=3$ and $r=0$, we obtain the respective terms

$$a_0^\mu, \qquad a_3^\mu. \tag{43}$$

The general term with $r=1$ is

$$a_0^{e_0}a_1^{e_1}a_2^{e_2}a_3^{e_3}, \qquad 3e_0+2e_1+e_2=\mu, \qquad e_1+2e_2+3e_3=2\mu. \tag{44}$$

In view of (5), an invariant ϕ must be unaltered by the substitution

$$a_0'=-a_3, \qquad a_3'=a_0, \qquad a_1'=a_2, \qquad a_2'=-a_1. \tag{45}$$

For $p=2$, the signs may be taken positive. For $p>2$, μ is even, and hence e_0, e_2 are both even or both odd. Further, we cannot have $e_0=e_3$ and $e_1=e_2$ in a term (44). Hence each term (44) of ϕ is accompanied by another term, not of type (44), having the same coefficient as the former. These new terms give all the terms of ϕ for which $r=2$, $s=1$ in (42).

In all the cases, viz., for $p^n=2^n$ $(n=1,\cdots,5)$, 3^n (n general), 5, 7, 11, 13, in which I have actually constructed an absolute *invariant* K of degree $\mu=p^n-1$, I find that the two terms (43) do not occur. Excluding those terms, we have the simpler* problem: *to construct an absolute invariant K whose terms are those given by* (44) *with suitable coefficients, and those derived by applying the substitution* $(a_0a_3)(a_1a_2)$.

* Since the exponent of a_3 is at most $\frac{2}{3}\mu$. By (41), the powers of δ_3 are complicated.

The results for the case $n = 1$ are very simple; all the terms just mentioned actually occur in K. *The cubic form with coefficients modulo p, a prime, has an absolute invariant K of degree $p-1$ containing $2k$ terms, where k is the number of partitions of $p-1$ into $3e_0 + 2e_1 + e_2$.* For $p = 3$, K is given by (14). For $p \neq 3$, the results relate to the notation (40) for the cubic form. Then, for $p = 2$, $K = a_1 + a_2$. For $p = 5, 7, 11, 13$, the expressions for K follow in that order. The first terms in the parentheses are the terms (44), arranged so that if two terms have different exponents to a_0, that with the lesser precedes, while, if the exponents of a_0 are equal the term with the lesser exponent to a_1 precedes. The second term in any parenthesis is derived from the first by the substitution $(a_0 a_3)(a_1 a_2)$.

$$(46)\quad (a_2^4+a_1^4)-(a_1a_2^2a_3+a_0a_1^2a_2)-(a_1^2a_3^2+a_0^2a_2^2)+(a_0a_2a_3^2+a_0^2a_1a_3),\quad GF[5].$$

$$(47)\quad \begin{aligned}&(a_2^6+a_1^6)+3(a_1a_2^4a_3+a_0a_1^4a_2)+3(a_1^2a_2^2a_3^2+a_0^2a_1^2a_2^2)+(a_1^3a_3^3+a_0^3a_2^3)\\&+3(a_0a_2^3a_3^2+a_0^2a_1^3a_3)+2(a_0a_1a_2a_3^3+a_0^3a_1a_2a_3)+(a_0^2a_3^4+a_0^4a_3^2),\quad GF[7].\end{aligned}$$

$$(48)\quad \begin{aligned}&(a_2^{10}+a_1^{10})-3(a_1a_2^8a_3+a_0a_1^8a_2)-3(a_1^2a_2^6a_3^2+a_0^2a_1^6a_2^2)+4(a_1^3a_2^4a_3^2+a_0^3a_1^4a_2^3)\\&+(a_1^4a_2^2a_3^4+a_0^4a_1^2a_2^4)-(a_1^5a_3^5+a_0^5a_2^5)+6(a_0a_2^7a_3^2+a_0^2a_1^7a_3)\\&+6(a_0a_1a_2^5a_3^3+a_0^3a_1^5a_2a_3)+5(a_0a_1^2a_2^3a_3^4+a_0^4a_1^3a_2^2a_3)-3(a_0a_1^3a_2a_3^5+a_0^5a_1a_2^3a_3)\\&+5(a_0^2a_2^4a_3^4+a_0^4a_1^4a_3^2)+4(a_0^2a_1a_2^2a_3^5+a_0^5a_1^2a_2a_3^2)-(a_0^2a_1^2a_3^6+a_0^6a_2^2a_3^2)\\&+(a_0^3a_2a_3^6+a_0^6a_1a_3^3),\qquad GF[11].\end{aligned}$$

$$(49)\quad \begin{aligned}&(a_2^{12}+a_1^{12})+5(a_1a_2^{10}a_3+a_0a_1^{10}a_2)+5(a_1^2a_2^5a_3^2+a_0^2a_1^8a_2^2)-6(a_1^3a_2^6a_3^3+a_0^3a_1^6a_2^3)\\&+6(a_1^4a_2^4a_3^4+a_0^4a_1^4a_2^4)+2(a_0^2a_2^6a_3^4+a_0^4a_1^6a_3^2)+2(a_1^5a_2^2a_3^5+a_0^5a_1^2a_2^5)+(a_1^6a_3^6+a_0^6a_2^6)\\&-3(a_0a_2^9a_3^2+a_0^2a_1^9a_3)+2(a_0a_1a_2^7a_3^3+a_0^3a_1^7a_2a_3)-3(a_0a_1^2a_2^5a_3^4+a_0^4a_1^5a_2^2a_3)\\&+3(a_0a_1^3a_2^3a_3^5+a_0^5a_1^3a_2^3a_3)-3(a_0a_1^4a_2a_3^6+a_0^6a_1a_2^4a_3)+4(a_0^2a_1a_2^4a_3^5+a_0^5a_1^4a_2a_3^2)\\&-3(a_0^2a_1^2a_2^2a_3^6+a_0^6a_1^2a_2^2a_3^2)+3(a_0^2a_1^3a_3^7+a_0^7a_2^3a_3^2)-6(a_0^3a_2^3a_3^6+a_0^6a_1^3a_3^3)\\&+2(a_0^3a_1a_2a_3^7+a_0^7a_1a_2a_3^3)+(a_0^4a_3^8+a_0^8a_3^4),\qquad GF[13].\end{aligned}$$

17. For a cubic form in the $GF[p^n]$, $n > 1$, the invariant K of degree $\mu = p^n - 1$ does not involve all the terms (44). Indeed, when the multiplicative constant is suitably chosen, K is identical* with K^p, so that a term (44) occurs in K only when

$$(50)\qquad [p^i e_0] + [p^i e_1] + [p^i e_2] + [p^i e_3] = \mu \quad (i = 1, \cdots, n-1),$$

where $[x]$ denotes the least positive (or zero) residue of x modulo μ. Inversely, *K contains every term* (44) *satisfying conditions* (50). In exhibiting an invariant, we enclose within a parenthesis terms derived as follows: first a term (44); then in turn its pth, (p^2)th, $\cdots$, (p^{n-1})th powers, when they are distinct

* The explanation of this property appears to lie in the fact noted at the end of § 12.

from the first term; finally, the terms derived from the preceding by the substitution $(a_0a_3)(a_1a_2)$, when they are distinct from the preceding.

18. For a cubic form in the $GF[2^n]$, the absolute invariant K of degree 2^n-1 contains exactly 2^n terms. We exhibit K for $n \leqq 5$, arranging the terms as explained in §§ 16, 17.

$$(a_2+a_1), \qquad GF[2]. \tag{51}$$

$$(a_2^3+a_1^3)+(a_0a_3^2+a_0^2a_3), \qquad GF[4]. \tag{52}$$

$$(a_2^7+a_1^7)+(a_0a_2^4a_3^2+a_0^2a_2a_3^4+a_0^4a_2^2a_3+a_0^2a_1^4a_3+a_0^4a_1a_3^2+a_0a_1^2a_3^4), \quad GF[8]. \tag{53}$$

$$\begin{aligned}&(a_2^{15}+a_1^{15})+(a_0a_2^{12}a_3^2+a_0^2a_2^9a_3^4+a_0^4a_2^3a_3^8+a_0^8a_2^6a_3+a_0^2a_1^{12}a_3+a_0^4a_1^9a_3^2+a_0^8a_1^3a_3^4\\&+a_0a_1^6a_3^8)+(a_0a_1^2a_2^8a_3^4+a_0^2a_1^4a_2a_3^8+a_0^4a_1^8a_2^2a_3+a_0^8a_1a_2^4a_3^2)+(a_0^5a_3^{10}+a_0^{10}a_3^5),\end{aligned} \tag{54}$$
$GF[16]$.

$$\begin{aligned}&(a_2^{31}+a_1^{31})+(a_0a_2^{28}a_3^2+a_0^2a_2^{25}a_3^4+a_0^4a_2^{19}a_3^8+a_0^8a_2^7a_3^{16}+a_0^{16}a_2^{14}a_3+a_0^2a_1^{28}a_3\\&+a_0^4a_1^{25}a_3^2+a_0^8a_1^{19}a_3^4+a_0^{16}a_1^7a_3^8+a_0a_1^{14}a_3^{16})+(a_0a_1^2a_2^{24}a_3^4+a_0^2a_1^4a_2^{17}a_3^8+a_0^4a_1^8a_2^3a_3^{16}\\&+a_0^8a_1^{16}a_2^6a_3+a_0^{16}a_1a_2^{12}a_3^2+a_0^4a_1^{24}a_2^2a_3+a_0^8a_1^{17}a_2^4a_3^2+a_0^{16}a_1^3a_2^8a_3^4+a_0a_1^6a_2^{16}a_3^8\\&+a_0^2a_1^{12}a_2a_3^{16})+(a_0^5a_2^{16}a_3^{10}+a_0^{10}a_2a_3^{20}+a_0^{20}a_2^2a_3^9+a_0^9a_2^4a_3^{18}+a_0^{18}a_2^8a_3^5+a_0^{10}a_1^{16}a_3^5\\&+a_0^{20}a_1a_3^{10}+a_0^9a_1^2a_3^{20}+a_0^{18}a_1^4a_3^9+a_0^5a_1^8a_3^{18}),\end{aligned} \tag{55}$$
$GF[32]$.

19. For the cubic form, with coefficients in the $GF[3^n]$,

$$a_0x^3+a_1x^2y+a_2xy^2+a_3y^3, \tag{56}$$

we can determine immediately the absolute invariant K of degree 3^n-1 in terms of the absolute invariant P, given by (9), and the discriminant Δ of (56), viz.,

$$P=(a_1^{3^n-1}-1)(a_2^{3^n-1}-1), \qquad \Delta=a_1^2a_2^2-a_0a_2^3-a_1^3a_3. \tag{57}$$

In fact, the invariant K is simply the reduced form of

$$K=\Delta^{(3^n-1)/2}-P+1. \tag{58}$$

The reduced form of K depends upon the character of the product

$$\pi\cdot\Delta^{3^{n-1}}, \qquad \pi\equiv\prod_{i=0}^{n-2}\Delta^{3^i}. \tag{59}$$

But in the $GF[3^n]$, we have

$$\Delta^{3^i}=(a_1a_2)^{2\cdot3^i}-a_0^{3^i}a_2^{3^{i+1}}-a_3^{3^i}a_1^{3^{i+1}}(i<n-1), \quad \Delta^{3^{n-1}}=(a_1a_2)^{2\cdot3^{n-1}}-a_0^{3^{n-1}}a_2-a_3^{3^{n-1}}a_1.$$

Now every term of the expanded product π is of degree

$$4\sum_{i=0}^{n-2}3^i=2d, \qquad d\equiv3^{n-1}-1.$$

Further, $\pi = (a_1 a_2)^d + \sigma$, where each term of σ is of degree higher than d in one of the letters a_1, a_2. Indeed, if we employ either of the last two terms of $\Delta^{3^{n-2}}$ the resulting term of the product π is of degree $\geqq 3^{n-1}$ in a_1 or a_2. Hence the statement follows by induction from $n-1$ to n.

It now follows that, in the complete product (59), a term is exactly of degree $3^n - 1$ if one of its factors is chosen from the last two terms of $\Delta^{3^{n-1}}$. Finally, in

$$\pi \cdot (a_1 a_2)^{2 \cdot 3^{n-1}} = (a_1 a_2)^{3^n - 1} + \sigma (a_1 a_2)^{2 \cdot 3^{n-1}}$$

every term, except the first, is of degree higher than $d + 2 \cdot 3^{n-1} \equiv 3^n - 1$ in one of the letters a_1, a_2, and hence admits of a reduction in total degree (by means of the equation $a^{3^n} = a$, satisfied by every mark of the field) from $2d + 4 \cdot 3^{n-1} = 2(3^n - 1)$ to $3^n - 1$. The exceptional first term cancels a term of $-P$. Hence *every term in the reduced form of the absolute* invariant* (58) *is of degree* $3^n - 1$.

In the expansion of (59) no two terms have the same set of exponents, so that there result 3^n distinct terms, no one identical with $a_j^{3^n-1}$ ($j = 1$ or 2). Hence *the reduced form of* (58) *contains exactly* $3^n + 1$ *terms*. For $n = 1$, they are given by (14); for $n = 2$ by (75). For $n \leqq 3$, I have verified that the terms of K obey the laws stated in § 17. As these (empirical) laws were formulated from another standpoint (and in fact prior to the investigation of the present case $p = 3$), we have independent evidence of the validity of the conjectures.

From (57) and (58) we deduce immediately the important relation

$$(\Delta^{(3^n+1)/2} - \Delta) K = 0. \tag{60}$$

Indeed, $\Delta P = 0$, so that

$$\Delta K = \Delta^{(3^n+1)/2} + \Delta, \qquad \Delta^{(3^n+1)/2} K = \Delta^{3^n} + \Delta^{(3^n+1)/2}, \qquad \Delta^{3^n} = \Delta.$$

20. The discriminant of the cubic form (40) in the $GF[p^n]$, $p \neq 3$, is

$$\Delta = -3a_1^2 a_2^2 + 4a_0 a_2^3 + 4a_1^3 a_3 - 6a_0 a_1 a_2 a_3 + a_0^2 a_3^2. \tag{61}$$

Now the invariants K, given by (46)-(49), satisfy the relation

$$(\Delta^{(p^n+1)/2} - \epsilon\Delta) K = 0, \qquad \epsilon = (-3)^{(p^n-1)/2}, \tag{62}$$

ϵ being $+1$ or -1 according as -3 is a square or a not-square in the $GF[p^n]$, $p > 3$. If we define Δ to be the function (61) multiplied by -3 (or by $-3\lambda^2$, λ any mark $\neq 0$), relation (62) is replaced by the simpler relation

$$(\Delta^{(p^n+1)/2} - \Delta) K = 0. \tag{63}$$

* Under a transformation of determinant D, the discriminant Δ is multiplied by D^6, so that $\Delta^{(3^n-1)/2}$ is an absolute invariant in the $GF[3^n]$.

THEOREM.* *If the indeterminate constant factor in the discriminant* Δ *of a cubic form in the* $GF[p^n]$, $p>2$, *be chosen so that the coefficient of* $a_1^2a_2^2$ *is a square in the field, then* Δ *and the absolute invariant* K *of degree* p^n-1 *satisfy the relation* (63).

21. Consider next invariants of the cubic form (56) in the $GF[3^n]$. Under transformation (3), the increments to a_2 and a_3 are $-ta_1$ and $ta_2+t^2a_1+t^3a_0$, respectively, while a_0 and a_1 are unaltered. The case $n=1$ was treated in § 9. Let here $n=2$. Then the coefficients of t and t^3 in $\phi'-\phi$ give the respective annihilators:†

$$
\begin{aligned}
&-a_1(2)+a_2(3)+a_0a_1^6(2^63)-a_1^5(2^73)+a_1^5a_2(2^83)-a_0^2a_1^3(2^33^2)\\
&-a_0a_1^5(2^43^2)+(a_0a_1^5a_2-a_1^7)(2^53^2)-a_1^7a_2(2^63^2)-a_1^7a_2^2(2^73^2)-a_0^3(3^3)\\
&-a_1^6(2^33^3)+a_1^6a_2^3(2^63^3)+a_0^3a_1^3(2^83^3)+a_0a_1^3(3^4)-a_1^5(23^4)+a_1^5a_2(2^33^4)\\
&-a_0a_1^3a_2^3(2^33^4)+a_1^5a_2^3(2^43^4)-a_1^5(a_0^4+a_2^4)(2^53^4)+a_0^3a_1^7(2^63^4)-a_0^3a_1^7a_2(2^73^4)\\
(64)\quad&+a_0a_1^3(2^83^4)+(a_0^2a_2^3-a_1^4a_2)(3^5)+(a_0a_1^2a_2^3-a_1^4a_2^2)(23^5)+(a_0^4a_1^4+a_1^4a_2^4)(2^33^5)\\
&+(a_0^3a_1^6a_2^2-a_0a_1^2)(2^63^5)-a_1^3a_2^3(3^6)-a_0^3a_1^5(2^23^6)-a_1^3a_2^6(2^33^6)\\
&+(a_0^3a_1^5a_2^3-a_1^3)(2^53^6)-a_0^6a_1^7(2^73^6)-a_1^3a_2^3(2^83^6)+(a_0a_2^6-a_0^3a_1^4)(3^7)\\
&+(a_0^3a_1^4a_2-a_1^2a_2^6)(23^7)+(a_1^2a_2^7-a_0^4a_1^2a_2^3+a_0a_1^8)(2^23^7)+(a_0^3a_1^4a_2^3-a_1^2)(2^33^7)\\
&+(a_0^4a_1a_2^3-a_0^3a_1^3a_2^2-a_0a_1^7-a_1a_2^7)(3^8);
\end{aligned}
$$

$$
\begin{aligned}
&-a_1^3(2^3)+a_0(3)-a_1^2(23)+a_1^2a_2(2^23)+a_0a_1^3(2^33)-a_1a_2(3^2)-a_1a_2^2(23^2)\\
&-a_0^2a_1^5(2^53^2)-a_0a_1^7(2^63^2)+(a_0a_1^7a_2-a_1)(2^73^2)-a_1a_2(2^83^2)+a_2^3(3^3)\\
(65)\quad&+a_0^3a_1^2(2^23^3)-a_1^3(2^53^3)+a_1^3a_2^3(2^83^3)+a_0^3a_1(3^4)-a_0^3a_1a_2(23^4)+a_0a_1^5(2^33^4)\\
&-a_1^7(2^33^4)+a_1^7a_2(2^43^4)-a_0a_1^5a_2^2(2^53^4)+a_1^7a_2^2(2^63^4)-(a_0^4a_1^7+a_1^7a_2^4)(2^73^4)\\
&+a_0^3a_1(2^83^4)+(a_0^3a_2^2-a_0a_1^4)(3^5)+(a_0a_1^4a_2-a_1^6)(23^5)+(a_0a_1^4a_2^3-a_1^6a_2^2)(2^33^5)
\end{aligned}
$$

* If, instead of the mere verification of this theorem for the computed invariants K, we had a direct proof, it would be of decided aid to the computation. We may compute K by means of (63) alone. For example, let $p^n=3^2$, and give K its necessary form (75) with the four parentheses multiplied by $1, m, s, r$, respectively. In $(\Delta^5-\Delta)K$, we readily find that the factor of a_3^8 is $(s-r)a_0a_1^8a_2^3-(s+m)a_1^7a_2^2$, while the terms independent of a_1 and a_3 are $(s-1)(a_0^5a_2^7-a_0a_2^3)$. Hence $-m=r=s=1$.

In general, the labor of determining the high power of Δ would be great. We may, however, employ only the first two terms of (61) and so readily compute the terms of (63) independent of a_3. We thereby get the coefficients of the terms of ϕ independent of a_3. The coefficients of the terms involving a_3 to the first power are then either known or else occur in the terms involving a_3^e $(e\geqq 2)$, in view of (45). The work with the annihilators is now greatly simplified. It suffices to find the coefficients of a_3^r, $\cdots$, a_3^3, a_3^2, in turn, where r is the greatest integer in $2/3(p^n-1)$. We need only a part of the annihilators; for instance, the derivative with respect to a_3 need not be of order higher than $r-2$.

† We have omitted the terms (2^i3^8), $i\geqq 1$, (2^i3^7), $i\geqq 4$, (2^i3^5), $i=2, 4, 5, 7, 8$ (which have long coefficients), since they have no effect upon a function ϕ satisfying (67), the binomial coefficients C_{8i} $(i=2, 4, 5)$ being multiples of 3.

$$+(a_1^6a_2^5+a_0^3a_1^8-a_0^4a_1^6a_2)(2^63^5)-a_0^6a_1(23^6)-a_1^5a_2^3(2^23^6)-a_0^5a_1^7(2^13^6)$$
$$-a_1^5a_2^6(2^53^6)+(a_0^3a_1^7a_2^3-a_1^5)(2^73^6)+(a_0^8a_2-a_1^4a_2^3)(3^7)+(a_0^4a_1^4+a_1^4a_2^4)(23^7)$$
$$+(a_0a_1^2a_2^6-a_0^3a_1^6)(2^23^7)+(a_0^3a_1^6a_2-a_1^4a_2^6)(2^33^7)$$
$$+(a_0^4a_1^3a_2-a_0^3a_1^5-a_0a_1a_2^6-a_1^3a_2^5)(3^8).$$

We may set

$$\phi=\sum_{i,j}^{0,\cdots,8}B_{ij}a_2^ia_3^j \qquad (B\text{'s functions of } a_0,\ a_1). \tag{66}$$

Since a_3^3 occurs only the first term of (64) and of (65), we get

$$B_{i8}a_1=0 \qquad (i=1,\cdots,8).$$

In view of these relations, a_3^7 occurs only in the first two terms of (64) and the first two terms of (65). The resulting sets of conditions react on each other and give the first two conditions (69) and

$$B_{i8}=0\ (i=1,\cdots,7), \qquad B_{i7}a_1=0\ (i=1,4,5,6,7,8),$$

together with $B_{88}a_0=0$. From the latter and $B_{88}a_1=0$ follows

$$B_{88}=k\pi, \qquad \pi\equiv(a_0^8-1)(a_1^8-1).$$

Hence on replacing ϕ by $\phi-kI$, when I is the absolute invariant (2), we have $B_{88}=0$. If both $B_{ij}a_0$ and $B_{ij}a_1$ vanish, then $B_{ij}=c\pi$, and hence $B_{ij}=0$ in view of (16).

We consider in turn the coefficients of $a_3^6,\cdots,a_3$ in (64) and (65) and finally the terms * independent of a_3 in (65). The process of writing down the two coefficients of a_3^r is simplified by holding in mind those of the relations (67) and (68) which have been found in the earlier steps. The two resulting sets of conditions react on each other very considerably, and some further simplifications result from the earlier conditions. The final results are as follows:

(67) $B_{i8}=0\,(i=1,\cdots,8)$, $B_{i7}=0\,(i=1,4,\cdots,8)$, $B_{i5}=0\,(i=2,3,4,5,7,8)$;

(68) $B_{i6}a_1=0\,(i=1,7,8)$, $B_{i4}a_1=0\,(i=5,7)$, $B_{i2}a_1=0\,(i=3,5,8)$;

together with the set of conditions, referred to as (69):

$$B_{08}=P_{27}a_1,\ B_{08}a_0=-B_{37}a_1^3,\ B_{07}=-B_{26}a_1,\ B_{37}=-B_{56}a_1,\ B_{27}=B_{46}a_1,$$
$$B_{37}a_0=-B_{66}a_1^3,\ B_{27}a_0=B_{56}a_1^3,\ B_{07}a_0=B_{36}a_1^3,\ B_{15}a_1=-B_{08}a_0^3,\ B_{65}a_1^3=B_{08},$$

* Those in (64) are not required in computing the invariants; they were determined directly for each invariant and found to vanish, thus giving a check.

$B_{65}=B_{84}a_1$, $B_{15}=-B_{27}a_0^3$, $B_{05}=B_{24}a_1$, $B_{44}a_1=-B_{37}a_0^3$, $B_{63}a_0=-B_{37}$,
$B_{84}a_1^3=B_{27}$, $B_{14}a_1+B_{65}a_0a_1^6+B_{07}a_0^3=0$, $B_{44}a_1^3=-B_{15}a_0$, $B_{07}=B_{64}a_1^3$,
$B_{05}a_0=B_{15}a_1^2-B_{34}a_1^3$, $B_{74}=B_{86}a_0^3$, $B_{64}=B_{76}a_0^3-B_{83}a_1$, $B_{54}=B_{66}a_0^3+B_{73}a_1$,
$B_{44}=B_{56}a_0^3$, $B_{34}=B_{46}a_0^3-B_{53}a_1$, $B_{24}=B_{36}a_0^3+B_{43}a_1-B_{66}a_1^6$,
$B_{14}=B_{26}a_0^3-B_{84}a_0a_1^6$, $B_{04}+B_{84}+B_{23}a_1-B_{16}a_0^3+B_{46}a_1^6=0$,
$B_{64}a_0a_1^6=B_{13}a_1+B_{06}a_0^3$, $B_{56}=B_{84}a_0$, $B_{46}=B_{74}a_0+B_{84}a_1^2$, $B_{36}=B_{64}a_0$,
$B_{26}=B_{54}a_0+B_{83}a_1^3$, $B_{16}=B_{44}a_0+B_{73}a_1^3$, $B_{06}+B_{86}=B_{34}a_0-B_{41}a_1^2+B_{63}a_1^3$,
$B_{76}=B_{24}a_0-B_{53}a_1^3-B_{08}a_0^3$, $B_{66}=B_{14}a_0-B_{24}a_1^2-B_{43}a_1^3-B_{05}a_1$,
$B_{04}a_0=B_{33}a_1^3+B_{26}a_0^3a_1^2-B_{56}a_1^8$, $B_{12}a_1=B_{05}a_0^3$, $B_{22}a_1=-B_{15}a_0^3$,
$B_{42}a_1=-B_{65}a_1^6-B_{08}a_1^3$, $B_{15}=-B_{72}a_1^3$, $B_{05}=-B_{62}a_1^3$, $B_{65}=B_{42}a_1^3$,
$B_{72}=B_{84}a_0^3$, $B_{62}=B_{81}a_1+B_{74}a_0^3$, $B_{52}=B_{64}a_0^3+B_{37}a_1^3-B_{71}a_1$, $B_{42}=B_{54}a_0^3+B_{27}a_1^3$,
$B_{32}=B_{51}a_1+B_{44}a_0^3$, $B_{22}=B_{34}a_0^3+B_{07}a_1^3-B_{41}a_1$, $B_{12}=B_{24}a_0^3$,
$B_{02}+B_{82}=B_{21}a_1-B_{72}a_0a_1^6+B_{14}a_0^3$, $B_{62}a_0a_1^6=B_{04}a_0^3+B_{84}a_0^3a_1^8-B_{11}a_1$,
$B_{54}=B_{82}a_0$, $B_{44}=B_{72}a_0$, $B_{34}=B_{62}a_0-B_{84}a_0^3a_1^2$, $B_{24}=B_{52}a_0-B_{81}a_1^3$,
$B_{14}=B_{42}a_0-B_{71}a_1^3$, $B_{04}+B_{84}=B_{32}a_0-B_{61}a_1^3-B_{27}a_1^5$, $B_{74}=B_{22}a_0+B_{51}a_1^3$,
$B_{64}+B_{22}a_1^2=B_{12}a_0+B_{41}a_1^3$, $B_{12}a_1^2=B_{02}a_0+B_{31}a_1^3$, $B_{53}+B_{81}a_0+B_{72}a_1=0$,
$B_{43}+B_{71}a_0+B_{62}a_1=0$, $B_{33}+B_{61}a_0-B_{71}a_1^2=0$, $B_{23}+B_{51}a_0+B_{80}a_1^3+B_{84}a_1^7=0$,
$B_{13}+B_{41}a_0-B_{51}a_1^2+B_{70}a_1^3=0$, $B_{03}+B_{83}+B_{31}a_0-B_{41}a_1^2+B_{60}a_1^3=0$,
$B_{73}+B_{21}a_0+B_{12}a_1-B_{50}a_1^3=0$, $B_{63}+B_{11}a_0-B_{21}a_1^2-B_{02}a_1-B_{40}a_1^3=0$,
$B_{01}a_0-B_{11}a_1^2-B_{30}a_1^3+B_{23}a_0^3a_1^2-B_{53}a_1^8+B_{04}a_0^3a_1+B_{24}a_0a_1^5=0$.

The general term of ϕ, in the notations of § 5, is

(70) $a_0^{e_0}a_1^{e_1}a_2^{e_2}a_3^{e_3}$, $\quad 3e_0+2e_1+e_2\equiv d$, $\quad e_1+2e_2+3e_3\equiv d \pmod 8$.

For use in the later cases, we first determine the absolute invariants, viz., those with $d\equiv 0$. If $B_{e_2e_3}a_1=0$, then $e_1\equiv 0$, and hence $B=0$ unless $e_3\equiv 2e_2 \pmod 8$. Hence by (68),

(71) $B_{16}=B_{86}=B_{54}=B_{74}=B_{32}=B_{82}=0$, $\quad B_{76}=ca_0^3(a_1^8-1)$, $\quad B_{52}=ka_0(a_1^8-1)$.

For B_{27}, $e_1=7$, $e_0=0$ or 8; but $a_0^8a_1^7$ does not occur in view of (16) since $B_{78}=0$. Hence $B_{27}=ra_1^7$, r a constant. Then $B_{08}=ra_1^8$, $B_{37}=-ra_0a_1^5$,

etc., by (69). But Δ^4 and Δ^8 are absolute invariants, Δ being the discriminant (57), and

$$\Delta^8 = a_3^8 a_1^8 + a_3^7(a_1^7 a_2^2 - a_0 a_1^5 a_2^3) + \cdots. \tag{72}$$

Hence if we replace ϕ by $\phi - r\Delta^8$, we may set $r = 0$. Hence, by (69)

$$B_{08}, B_{27}, B_{37}, B_{56}, B_{46}, B_{66}, B_{15}, B_{65}, \quad B_{84}, B_{44}, B_{73}, B_{22}, B_{42}, B_{72}, B_{51} \text{ all vanish.} \tag{73}$$

Now ϕ must be unaltered by substitution (16) and hence by $(a_0 a_3)(a_1 a_2)$, since e_0 and e_2 are both even or both odd, by (70). Thus, from (71), $B_{83} = c a_0^6 a_1^7$. Note that from $B_{07} = -c a_0^6 a_1^3$, we can conclude initially only that $B_{36} = -c a_0^7 + \rho a_0^7 a_1^8$, but subsequently that $\rho = 0$ since $B_{87} = 0$. In this way we readily find that conditions (69) are all satisfied if and only if, in addition to (71) and (73), the following relations hold:

$$B_{83} = c a_0^6 a_1^7,\ B_{03} = -c a_0^6 a_1^7,\ B_{81} = k a_0^2 a_1^5,\ B_{01} = -k a_0^2 a_1^5,\ B_{64} = -c a_0^8,$$

$$B_{06} = -c a_0^4 a_1^6,\ B_{07} = -c a_0^6 a_1^3,\ B_{36} = -c a_0^7,\ B_{14} = c a_0 a_1^2,\ B_{21} = c a_0^4 a_1,$$

$$B_{26} = c a_0^6 a_1^2,\ B_{24} = -k a_0^2,\ B_{02} = -k a_0^4 a_1^2,\ B_{05} = -k a_0^2 a_1,\ B_{12} = -k a_0^5,$$

$$B_{62} = k a_0^2 a_1^6,\ B_{71} = -k a_0 a_1^7,\ k = c,\ B_{41} = B_{31} = B_{43} = 0,\ B_{34} = c a_0^3 a_1^6,$$

$$B_{63} = c a_0^4 a_1^3,\ B_{53} = -c a_0^3 a_1^5,\ B_{23} = w a_1^3 \ (w \text{ new parameter}),\ B_{30} = w a_0^3 a_1^2,$$

$$B_{04} = -w a_1^4,\ B_{40} = -w a_0^4,\ B_{11} = -(k+w) a_0^3 a_1^3,\ B_{33} = -(k+w) a_0 a_1,$$

$$B_{61} = w a_1,\ B_{10} = w a_0 a_1^6,\ B_{20} = B_{50} = B_{60} = B_{70} = B_{13} = 0,$$

$$B_{80} = l(a_1^8 - 1) - w \ (l \text{ new parameter}),\ B_{00} = -(w+l) a_1^8 + \text{constant}.$$

For convenience, we take the constant in B_{00} (the absolute term of ϕ) to be l. Then there are three independent parameters w, l and $c = k$. The coefficient of the latter is the negative of the absolute invariant (of degrees 24, 16, 8):

$$\begin{aligned} W = {} & a_3^7 a_0^6 a_1^3 + a_3^6\{-a_0^3(a_1^8 - 1) a_2^7 + a_0^7 a_2^3 - a_0^6 a_1^2 a_2^2 + a_0^4 a_1^6\} + a_3^5 a_0^2 a_1 \\ & + a_3^4(a_0^6 a_2^6 - a_0^3 a_1^6 a_2^3 + a_0^2 a_2^2 - a_0 a_1^2 a_2) + a_3^3\{-a_0^6 a_1^7(a_2^8 - 1) - a_0^4 a_1^3 a_2^6 + a_0^3 a_1^5 a_2^5 + a_0 a_1 a_2^3\} \\ & + a_3^2\{-a_0^2 a_1^6 a_2^6 - a_0(a_1^8 - 1) a_2^5 + a_0^5 a_2 + a_0^4 a_1^2\} \\ & + a_3\{-a_0^2 a_1^5(a_2^8 - 1) + a_0 a_1^7 a_2^7 - a_0^4 a_1 a_2^2 + a_0^3 a_1^3 a_2\}. \end{aligned} \tag{74}$$

The coefficient of l is the invariant P, given by (57). The negative of the coefficient of w is the absolute invariant of degree 8 (see § 19):

$$K = a_3^4 a_1^4 + a_3^3(a_0 a_1 a_2^3 - a_1^3 a_2^2) + a_3(a_0^3 a_1^3 a_2 - a_1 a_2^6) + a_2^8 + a_0^4 a_2^4 - a_0^3 a_1^2 a_2^3 - a_0 a_1^6 a_2 + a_1^8. \tag{78}$$

Now $K = \Delta^4 - P + 1$, in agreement with (58). *Hence the absolute invariants are the linear functions of* $I, P, W, \Delta^4, \Delta^8$. The system should be closed also under multiplication. In verification, we note the relations

$$(76) \qquad \begin{gathered} I^2 = I,\ IP = I,\ IW = 0,\ I\Delta = 0,\ P^2 = P,\ PW = 0, \\ P\Delta = 0,\ W^2 = W,\ W\Delta^4 = W. \end{gathered}$$

Let next $d \equiv 1 \pmod 8$. Then $B_{e_2e_3}a_1 = 0$ implies $B = 0$ unless $e_3 \equiv 2e_2 + 3 \pmod 8$. Hence the B_{ij} in (68) all vanish. For B_{27}, $e_1 = 0$ or 8, $e_0 = 5$; but $a_0^5 a_1^8$ does not occur in view of (16) since $B_{85} = 0$. Thus $B_{27} = ca_0^5$. Also $B_{84} = ra_0^5 a_1^5$. But one of relations (69) is $B_{84}a_1^3 = B_{27}$. Hence $B_{27} = B_{84} = 0$. In view of relations (69) and the invariance* of ϕ under (16), we readily find that the only non-vanishing B_{ij} are the sixteen for which i, j range over the exponents of a_2, a_3 in the following invariant, the value of B_{ij} being the product of a fixed parameter l by the coefficient of $a_2^i a_3^j$:

$$(77) \qquad \begin{aligned} &a_3^7 a_0^3 a_1^4 + a_3^6(a_0^4 a_1 a_2^3 - a_0^3 a_1^3 a_2^2 - a_0 a_1^7) + a_3^4(a_0^3 a_1 a_2^6 - a_0^6 a_1^3 a_2) \\ &+ a_3^3(-a_0^3 a_2^8 - a_0 a_1^4 a_2^6 - a_0^7 a_2^4 + a_0^6 a_1^2 a_2^3 - a_0^4 a_1^6 a_2 + a_0^3 a_1^8) + a_2^3(-a_0 a_1^3) \\ &+ a_3(a_0^6 a_2^7 + a_0^3 a_1^6 a_2^4 + a_0^2 a_2^3), \end{aligned}$$

which equals $-E^3$, E being given by (10). By (82), $E^3 = \Delta E$.

For $d \equiv 3 \pmod 8$, we conclude that E is the only invariant. Indeed, the operation of cubing is uniquely reversible in the $GF[9]$. Similarly, the case $d \equiv 2$ reduces to the case $d \equiv 8$, and $d \equiv 7$ to $d \equiv 5$.

For $d \equiv 6 \pmod 8$, we may set $B_{08} = ka_0^6 a_1^6$. Then $B_{27} = ka_0^6 a_1^5$, etc. If we replace ϕ by $\phi + k\Delta W$, where

$$(78) \qquad \Delta W = a_3^8(-a_0^6 a_1^6) + a_3^7(a_0^7 a_1^5 a_2^3 - a_0^6 a_1^5 a_2^2 - a_0^4 a_1) + \cdots,$$

we may set $k = 0$. The resulting invariant is easily found to be a linear function of Δ and Δ^5. Hence for $d \equiv 2$, the invariants are the linear functions of Δ^3, Δ^7 and $\Delta^3 W^3 \equiv \Delta^3 W$.

For $d \equiv 4$, we have $B_{27} = da_0^4 a_1^3$, $B_{08} = da_0^4 a_1^4$, $B_{37} = -da_0^5 a_1$, $B_{07} = da_0^2 a_1^7$, etc. We replace ϕ by $\phi - d\Delta^2 W$, where

$$(79) \qquad \Delta^2 W = a_3^8 a_0^4 a_1^4 + a_3^7(-a_0^5 a_1 a_2^3 + a_0^4 a_1^3 a_2^2 + a_0^2 a_1^7) + \cdots,$$

and hence may set $d = 0$. The resulting invariant is readily found to be a linear function of Δ^2 and Δ^6.

* Here a more exacting condition than for $d \equiv 0$. Thus $B_{21} = ma_0 a_1^2$ yields $m = 0$.

For $d \equiv 5$, we have $B_{27} = ra_0a_1^4$, $B_{08} = ra_0a_1^5$, $B_{37} = -ra_0^2a_1^2$, etc. We replace ϕ by $\phi - r\Delta^3 E$, where

$$\Delta^3 E = a_3^8 a_0 a_1^5 + a_3^7(a_0 a_1^4 a_2^2 - a_0^2 a_1^2 a_2^3) + \cdots, \tag{80}$$

and easily find that the resulting invariant vanishes identically. Then for $d \equiv 7$, the only invariant is $(\Delta^3 E)^3 \equiv \Delta^2 E$, by (82).

THEOREM. *The cubic form* (57) *in the* $GF[9]$ *has exactly* 18 *linearly independent invariants. These may be taken to be*

$$I, P, \Delta^i\ (i = 1, \cdots, 8), \qquad \Delta^j W, \Delta^j E\ (j = 0, 1, 2, 3). \tag{81}$$

The product of any two invariants can be reduced to a linear function of these 18 *invariants by the application of relations* (76) *and*

$$IE = PE = 0, \quad WE = E, \quad E^2 = \Delta W, \quad \Delta^4 E = E, \quad \Delta^9 = \Delta. \tag{82}$$

22. The invariants I and P of the cubic form in the $GF[3^n]$ can be expressed in terms of a single invariant Γ. Thus

$$\Gamma = I + P, \qquad I = \Gamma - \Gamma^2, \qquad P = \Gamma^2. \tag{83}$$

For $n = 1$, $W = E^2$ by (19). For $n = 2$, we have, by (76), (82),

$$W = W\Delta^4 = E^2\Delta^3.$$

Hence we have proved, for $n = 1$ and $n = 2$, the following

THEOREM. *The invariants of the cubic form in the* $GF[3^n]$ *are all rational integral functions of the three fundamental invariants* $\Gamma = I + P$, E, *and the discriminant* Δ, *where* I, P, E, *are defined in* §§ 1, 3, 4.

23. For the cubic form in the $GF[2^n]$, the power 2^{n-1} of the discriminant $a_0^2a_3^2 + a_1^2a_2^2$ gives the invariant

$$D = a_0a_3 + a_1a_2. \tag{84}$$

The eliminant E (§ 4) is $a_0a_3\pi$, where for $n = 1, 2, 3$, respectively, π equals

$$\Sigma a_i, \qquad \Sigma a_i^3 + D(a_0 + a_3), \qquad \Sigma a_i^7 + \lambda + \mu,$$

λ being the expression in the second parenthesis of (53), and

$$\mu = a_1a_2^3a_3 + a_0a_1a_2^2a_3^3 + a_0^3a_1a_2^3 + a_0a_1^5a_2 + a_0^3a_1^2a_2a_3 + a_1^3a_2a_3^3.$$

Thus $a_0a_3\mu = a_1a_2\lambda$. Hence, in each case, $E = DK$, K being given in § 18.

For $n = 1$ *or* 2, *every invariant is an absolute invariant.* This is evident for $n = 1$. For $n = 2$, we add the congruences (70), the modulus now being

3; there results $0 \equiv d \pmod 3$. By use of the annihilators, I have found the invariants

$$(85) \qquad R_{n=1} = a_0 + a_3 + a_0a_1 + a_0a_2 + a_1a_2 + a_1a_3 + a_2a_3,$$

$$(86) \quad R_{n=2} = a_0 + a_3 + a_0a_1^3 + a_0a_2^3 + a_0a_3^3 + a_0^3a_3 + a_1^3a_3 + a_2^3a_3 + a_0^2a_1a_2 + a_1a_2a_3^2 + a_0a_1a_2a_3,$$

and have proved that, for $n = 1, 2$, every invariant is a rational integral function of D, K, R. In particular, an expression for (2) is

$$(87) \qquad I = (D^\mu - 1)(K^\mu - 1)(R^\mu - 1), \qquad \mu = 2^n - 1.$$

For $n = 1$, every invariant is a linear function of D, K, R, DK, DR. Any relation between these follows from $KR = 0, D^2 = D, K^2 = K, R^2 = R$.

For $n = 2$, the 13 linearly independent invariants may be taken to be

$$(88) \qquad K, \quad D^i, \quad R^i, \quad D^iR, \quad D^iK \qquad (i = 1, 2, 3).$$

Any relation between these (absolute) invariants follows from

$$(89) \quad KR = 0, \quad K^2 = K, \quad (DR)^2 = DR, \quad D^4 = D, \quad R^4 = R.$$

For a proof that D, R, K are independent, see §§ 27, 28.

24. For the cubic form (40) with the integral coefficients modulo 5, the only invariant with $d \equiv 3 \pmod 4$ is the eliminant, given by § 4,

$$(90) \quad E = a_3^3(2a_0^2a_2 - 2a_0a_1^2) + a_3^2(2a_0a_1a_2^2 - 2a_0^3a_1) + a_3(a_0a_2^4 + 2a_0^3a_2^2 - 2a_0^2a_1^2a_2 - a_0a_1^4);$$

while ΔE is the only invariant with $d \equiv 1$. The only (absolute) invariants with $d \equiv 0$ are the linear functions of $I, \Delta^2, \Delta^4, K, \Delta^2K$, where K is given by (46). The only invariants with $d \equiv 2$ are the linear functions of $\Delta, \Delta^3, \Delta K$.

The ten linearly independent invariants of the cubic form (40) in the $GF[5]$ may be taken to be

$$(91) \qquad I, \ E, \ \Delta E, \ K, \ \Delta K, \ \Delta^2K, \ \Delta, \ \Delta^2, \ \Delta^3, \ \Delta^4.$$

Any relation between these follows from

$$(92) \quad \begin{gathered} I^2 = I, \quad IE = I\Delta = IK = 0, \quad E^2 = \Delta - \Delta^3 - \Delta K, \quad \Delta^2E = -E, \\ KE = -E, \quad K^2 = K - \Delta^2 + \Delta^4, \quad \Delta^3K = -\Delta K, \quad \Delta^5 = \Delta. \end{gathered}$$

25. For the cubic form (40) with integral coefficients modulo 7, the non-absolute invariants have $d \equiv 3 \pmod 6$. Indeed, if we add the congruences $\pmod 6$, analogous to (70), we find that $0 \equiv d \pmod 3$.

The only invariants with $d \equiv 3 \pmod 6$ are the linear functions of E, ΔE, and $\Delta^2 E$, where the eliminant E is (end of § 4)

$$(93)\quad \begin{aligned} E = {} & a_3^5(3a_0^3) + a_3^4(2a_0a_1^3 - 3a_0^2a_1a_2) + a_3^3(a_0^2a_2^3 + a_0a_1^2a_2^2 - 3a_0^5) \\ & + a_3^2(5a_0a_1a_2^4 + 3a_0^4a_1a_2 - a_0^3a_1^3) + a_3(a_0a_2^6 - 2a_0^3a_2^3 - a_0^3a_1^2a_2^2 + 2a_0^2a_1^4a_2 - a_0a_1^6). \end{aligned}$$

The absolute invariants are functions of Δ, K, given by (47), and Q:

$$(94)\quad \begin{aligned} Q = {} & a_3^6(2a_0^2) + a_3^5(6a_0a_1a_2 + 3a_1^3) + a_3^4(a_0a_2^3 + a_1^2a_2^2) + a_3^3(2a_1a_2^4 + 3a_0^3a_1a_2 + 4a_0^2a_1^3) \\ & + a_3^2(2a_2^6 + 4a_0^3a_2^3 + 3a_0^2a_1^2a_2^2 + a_0a_1^4a_2 - a_1^6 + 2a_0^6 - 2) \\ & + a_3(a_0^2a_1a_2^4 - a_0a_1^3a_2^3 + 2a_1^5a_2^2 - a_0^5a_1a_2 + a_0^4a_1^3) \\ & - a_0^2a_2^6 + 2a_0a_1^2a_2^5 + 3a_0^3a_2^3 + a_0^4a_1^2a_2^2 + 2a_0^3a_1^4a_2 + 2a_0^2a_1^6 - 2a_0^7. \end{aligned}$$

The sixteen linearly independent invariants of the cubic form (40) *in the* $GF[7]$ *may be taken to be*

$$(95)\quad E,\ \Delta E,\ \Delta^2 E,\ Q,\ Q^2,\ Q^3,\ \Delta^i\ (i = 1, \cdots, 6),\ \Delta^j K\ (j = 0, 1, 2, 3).$$

Any relation between these follows from

$$(96)\quad \begin{aligned} & \Delta^3 E = E,\ EQ = 0,\ EK = -E,\ Q^4 = 3\Delta^5 - 3\Delta^2 - Q,\ QK = 0,\ \Delta^7 = \Delta, \\ & \Delta^4 K = \Delta K,\ \Delta Q = 2\Delta^3 - 2\Delta^6,\ K^2 = K + \Delta^6 + \Delta^3,\ E^2 = 2\Delta K - 2\Delta^4 - 2\Delta. \end{aligned}$$

The invariant (2) has the following expression:

$$(97)\quad I = Q^3 + \Delta^3 K - K + 2\Delta^6 - 3\Delta^3 + 1.$$

26. The following tables give a complete set of (non-equivalent) canonical forms of binary quadratic forms in the $GF[p^n]$ and the values of the independent invariants for each canonical form:

$p > 2$ (invariants, § 15).

Canonical.	Δ	J
$x^2 - \nu y^2$	ν	0
xy	1	0
x^2	0	-2
νx^2	0	0
Vanishing	0	-1

$p = 2$ (invariants, § 11).

Canonical.	a_1	I	K
$x^2 + xy + cy^2$	1	0	1
xy	1	0	0
x^2	0	0	0
Vanishing	0	1	0

Here ν denotes a particular not-square in the $GF[p^n]$, $p > 2$; while c is a particular solution of $c + c^2 + \cdots + c^{2^{n-1}} = 1$ in the $GF[2^n]$.

27. As a complete set of (non-equivalent) canonical cubic forms in the $GF[p^n]$, we may take

$$\beta C,\ \beta x Q,\ \beta xy(x+y),\ x^2y,\ \beta x^3,\ \text{vanishing form},$$

where C and Q are any particular irreducible cubic and quadratic forms, e. g., $C = x^3 - xy^2 + \tau y^3$, for a suitable value of τ, and (§ 26)

$$Q = x^2 - \nu y^2 \quad (p > 2), \qquad Q = x^2 + xy + cy^2 \quad (p = 2);$$

while $\beta = 1$ if $p^n = 3^n$ or $3l+2$; $\beta = 1, \epsilon, \epsilon^2$, if $p^n = 3l+1$, ϵ being a primitive root of the field.

$GF[2]$ (invariants, § 23).

Canonical.	D	R	K
$x^3 - xy^2 + y^3$	1	0	1
$x(x^2 + xy + y^2)$	1	0	0
$xy(x+y)$	1	1	0
x^2y	0	0	1
x^3	0	1	0
Vanishing	0	0	0

$GF[2^2]$, $i^2 \equiv i+1 \pmod 2$ (§ 23).

Canonical.	D	R	K
$\beta(x^3 - xy^2 + y^3)$	β^2	0	β
$\beta x(x^2 + xy + iy^2)$	$i\beta^2$	$i^2\beta$	$i^2\beta$
$\beta xy(x+y)$	β^2	0	0
x^2y	0	0	1
βx^3	0	β	0
Vanishing	0	0	0

$GF[3^n]$, $n = 1, 2$ (§ 22).

Canonical.	Δ	E	Γ
$x^3 - xy^2 + y^3$	1	1	0
$x(x^2 - \nu y^2)$	ν^3	0	0
$xy(x+y)$	1	0	0
x^2y	0	0	0
x^3	0	0	1
Vanishing	0	0	-1

$GF[5]$ (§ 24).

Canonical.	Δ	K	I	E
$x^3 - xy^2 + 2y^3$	2	-1	0	1
$x(x^2 - 2y^2)$	-1	0	0	0
$xy(x+y)$	2	2	0	0
x^2y	0	1	0	0
x^3	0	0	0	0
Vanishing	0	0	1	0

28. An inspection of the tables §§ 26, 27 shows that all the invariants given are necessary to characterize the canonical forms (since if one invariant is omitted, the others have equal values for some two canonical forms), and that no invariant is a rational function of the others (since any one takes different values when the others are equal). The only exception to these statements is the case of invariant E of the cubic form in the $GF[5]$; the characterization is complete without E; but the independence of E follows from the weights d (§ 24).

THE UNIVERSITY OF CHICAGO,
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PROJECTIVE DIFFERENTIAL GEOMETRY OF CURVED SURFACES*

(FIRST MEMOIR)

BY

E. J. WILCZYNSKI

§1. *The simultaneous solutions of two linear homogeneous partial differential equations of the second order with two independent variables.*

Consider a system of partial differential equations of the form

$$\begin{aligned} Ay_{uu} + By_{uv} + Cy_{vv} + Dy_u + Ey_v + Fy = 0, \\ A'y_{uu} + B'y_{uv} + C'y_{vv} + D'y_u + E'y_v + F'y = 0, \end{aligned} \tag{1}$$

where

$$y_u = \frac{\partial y}{\partial u}, \qquad y_v = \frac{\partial y}{\partial v}, \qquad y_{uu} = \frac{\partial^2 y}{\partial u^2}, \quad \text{etc.}, \tag{2}$$

and where $A, B, \cdots, F'$ are analytic functions of u and v.

If the determinant $AC' - A'C$ does not vanish identically, (1) may be written in the form

$$\begin{aligned} y_{uu} &= ay_{uv} + by_u + cy_v + dy, \\ y_{vv} &= a'y_{uv} + b'y_u + c'y_v + d'y. \end{aligned} \tag{3}$$

If the determinant $AC' - A'C$ is equal to zero, while $BC' - B'C$ does not vanish, we may write (1) in the form

$$\begin{aligned} y_{uu} &= \alpha y_{vv} + \beta y_u + \gamma y_v + \delta y, \\ y_{uv} &= \beta' y_u + \gamma' y_v + \delta' y, \end{aligned} \tag{4}$$

provided that A and A' do not vanish simultaneously. If they do, i. e., if $A = A' = 0$, while $BC' - B'C \neq 0$, we obtain from (1) a system of the same form as (4) except for an interchange of u and v, with the further specialization that $\alpha = 0$. Finally, if

$$AC' - A'C = BC' - B'C = 0,$$

one of the equations deducible from (1) reduces to the first order.

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Transform (4) by putting

$$u = r, \qquad v = r + s,$$

and let

$$y_r = \frac{\partial y}{\partial r}, \qquad y_s = \frac{\partial y}{\partial s}, \quad \text{etc.}$$

The system (4) becomes

$$y_{rr} - 2y_{rs} + y_{ss} = \alpha y_{ss} + \beta(y_r - y_s) + \gamma y_r + \delta y,$$
$$y_{rs} - y_{ss} = \beta'(y_r - y_s) + \gamma' y_s + \delta' y,$$

which equations may be solved for y_{rr} and y_{ss}. By such a transformation of the independent variables (4) may therefore be reduced to the form (3).

Unless, therefore, there exists a linear combination of the two equations (1) *which is of the first order only, system* (1) *may be reduced to the form* (3), *a reduction which may involve a linear transformation of the independent variables.*

Consider a system of form (3): y_{uu} and y_{vv} are expressed in terms of y, y_u, y_v, y_{uv}. Differentiation of the first of the two equations with respect to u will express y_{uuu} in terms of the same four quantities and of y_{uuv}. Similarly y_{vvv} will depend upon y_{uvv} as well as upon y, y_u, y_v, y_{uv}. In general, however, y_{uuv} and y_{uvv} may themselves be expressed in terms of y, y_u, y_v, y_{uv}. In fact differentiation of the first equation of (3) with respect to v, and of the second with respect to u will give

$$(5) \qquad \begin{aligned} y_{uuv} - a y_{uvv} &= (a_v + b) y_{uv} + c y_{vv} + b_v y_u + (c_v + d) y_v + d_v y, \\ -a' y_{uuv} + y_{uvv} &= (a'_u + c') y_{uv} + b' y_{uu} + (b'_u + d') y_u + c'_u y_v + d'_u y, \end{aligned}$$

whence follows the truth of the above statement provided that

$$(6) \qquad aa' - 1 \neq 0$$

In this case, clearly all higher derivatives of y may be expressed in the form

$$(7) \qquad \alpha y + \beta y_u + \gamma y_v + \delta y_{uv},$$

where α, β, γ, δ are, for every such derivative, perfectly definite functions of u and v. It is to be noticed, however, that of the five derivatives of the fourth order, three may be computed in two different ways; for example

$$y_{uuuv} = \frac{\partial}{\partial u}(y_{uuv}) = \frac{\partial}{\partial v}(y_{uuu}.)$$

Therefore, in order that the eight expressions of form (7) for the five fourth derivatives may be consistent, three conditions must be satisfied. It is not difficult to see that, if these three so-called *integrability conditions* are satisfied, the expression of form (7) for a derivative of any higher order will be unique.

Let y be a solution of system (3), developable in powers of $u-u_0$ and $v-v_0$, and denote by $c_1, \cdots, c_4$ the values of y, y_u, y_v, y_{uv} for $u=u_0, v=v_0$. Then y may be expressed in the form

$$y = c_1 + c_2(u-u_0) + c_3(v-v_0) + c_4(u-u_0)(v-v_0)$$
$$+\tfrac{1}{2}(d_0c_1+c_0c_2+b_0c_3+a_0c_4)(u-u_0)^2+\tfrac{1}{2}(d_0'c_1+c_0'c_2+b_0'c_3+a_0'c_4)(v-v_0)^2+\cdots,$$

where a_0, b_0, etc. are the values of $a, b, c, \cdots$ for $u=u_0, v=v_0$. All of the coefficients of this development will be linear combinations of $c_1, \cdots, c_4$. We may write, therefore,

$$y = c_1y' + c_2y'' + c_3y^{(3)} + c_4y^{(4)},$$

where $y', \cdots, y^{(4)}$ are holomorphic functions of $u-u_0$ and $v-v_0$. The most general analytic solution of (3) can, therefore, contain only four arbitrary constants which enter linearly.

The integrability conditions are of the form

$$\alpha y_{uv} + \beta y_u + \gamma y_v + \delta y = 0. \tag{8}$$

They may be satisfied *identically*, so that

$$\alpha = \beta = \gamma = \delta = 0.$$

In that case the solution just indicated in a formal manner actually contains four arbitrary constants. If the integrability conditions are not satisfied identically, they represent additional differential equations which y must verify. The number of arbitrary constants in the general solution of (3) is reduced at least by one for every independent non-identically satisfied integrability condition. The above condition (8), for example, requires that

$$\alpha_0c_4 + \beta_0c_2 + \gamma_0c_3 + \delta_0c_1$$

shall be equal to zero, where $\alpha_0 \cdots \delta_0$ are the values of $\alpha \cdots \delta$ for $u=u_0, v=v_0$. This clearly reduces the number of arbitrary constants to three.

If the integrability conditions are satisfied identically, the most general analytic solution of (3) can depend, therefore, upon only four arbitrary constants which enter linearly. Such an analytic solution actually exists if the coefficients of (3) are holomorphic in the vicinity of $u=u_0, v=v_0$. This may be proved by the method which, since the time of CAUCHY, has always been employed for such investigations.* Of course, it may seem desirable to prove the existence of just a four-fold infinity of solutions for a system of form (3), for which $aa'-1$ does not vanish and whose integrability conditions are satisfied identically under more general assumptions. For our present purpose, it suffices to restrict our considerations to systems of form (3) with analytic coefficients.

*Cf. for example GOURSAT, *Leçons sur l'intégration des équations aux dérivées partielles du premier ordre*, Paris, 1891, chapter I.

We proceed to consider the case

$$aa' - 1 = 0. \tag{9}$$

If we multiply both members of the second equation of (5) by a and add to the members of the first, taking account moreover of (3), we find

$$\begin{aligned}[a_v + b + a(a'_u + c') + a^2 b' + ca']y_{uu} + [b_v + a(b'_u + d') + abb' + cb']y_u \\ + [c_v + d + ac'_u + acb' + cc']y_v + [d_v + ad'_u + adb' + cd']y = 0.\end{aligned} \tag{10}$$

Unless all of the coefficients of this equation are zero, it becomes possible, by means of it, to reduce every expression of the form $\alpha y + \beta y_u + \gamma y_v + \delta y_{uv}$ to one involving only three terms, so that the general analytic solution of (3) could depend upon no more than three arbitrary constants. If, on the other hand, all of the coefficients of (10) are equal to zero, one of the partial derivatives of each order will remain arbitrary, and the general solution will contain an infinite number of arbitrary constants. This solution actually exists. The two equations are then said to be in *involution*;* the system is *involutory.*

We may recapitulate as follows. 1. *The system* (3) *of simultaneous linear differential equations has precisely four linearly independent solutions if* $aa' - 1$ *is different from zero and if the integrability conditions are identically satisfied.*

2. *If* $aa' - 1$ *vanishes, and the integrability conditions are satisfied, while not all of the coefficients of* (10) *are zero, the system* (3) *has less than four linearly independent solutions.*

3. *If, however, the first two conditions of No.* 2 *are satisfied, and if besides all of the coefficients of* (10) *are equal to zero, the general solution of system* (3) *contains an infinite number of arbitrary constants. The system is an involutory one.*

§ 2. *Geometric interpretation. The integrating surface.*

If $aa' - 1$ is different from zero and if all of the integrability conditions are satisfied identically, system (3) has just four linearly independent solutions. Denote four such solutions by y', y'', $y^{(3)}$, $y^{(4)}$. We shall have

$$y^{(k)} = f^{(k)}(u, v) \qquad (k = 1, 2, 3, 4).$$

Interpret $y^{(k)}$ as the homogeneous coördinates of a point P_y in space. As u and v assume all of their values, P_y will describe a surface S_y an *integrating surface* of the system. *This surface cannot degenerate into a curve.* For if

*Cf. GOURSAT, *Leçons sur l'intégration des équations aux dérivées partielles du second ordre à deux variables indépendantes*, Paris, 1896–98, vol. 2, chap. 6, where a more general case is treated according to BIANCHI.

S_y were to degenerate in that way, the ratios of $y', y'', y^{(3)}, y^{(4)}$, would become functions of a single variable $t = \phi(u, v)$. In that case $y', \cdots, y^{(4)}$ would be solutions of some linear partial differential equation of the first order, say

$$\alpha y_u + \beta y_v + \gamma y = 0, \tag{11}$$

besides satisfying (3). As a matter of fact let

$$\alpha w_u + \beta w_v = 0$$

be the partial differential equation satisfied by $t = \phi(u, v)$. Then if

$$\frac{y'}{y^{(4)}}, \quad \frac{y''}{y^{(4)}}, \quad \frac{y^{(3)}}{y^{(4)}},$$

were functions of t alone, they would satisfy the same equation, so that

$$\alpha(y_u^{(k)} y^{(4)} - y^{(k)} y_u^{(4)}) + \beta(y_v^{(k)} y^{(4)} - y^{(k)} y_v^{(4)}) = 0 \qquad (k = 1, 2, 3),$$

or

$$\alpha y_u^{(k)} + \beta y_v^{(k)} - \frac{\alpha y_u^{(4)} + \beta y_v^{(4)}}{y^{(4)}} y^{(k)} = 0 \qquad (k = 1, 2, 3).$$

Put

$$-\frac{\alpha y_u^{(4)} + \beta y_v^{(4)}}{y^{(4)}} = \gamma,$$

and it becomes clear that $y', y'', y^{(3)}, y^{(4)}$ would satisfy the same equation of form (11). Differentiate (11) with respect to u and v. If the two equations thus obtained coincide with the two equations of (3), $aa' - 1$ would be equal to zero, contrary to our assumption. (11) would be an intermediary integral of (3) whose general solution would depend upon an arbitrary function. If, on the other hand, the two equations obtained from (11) by differentiation were not identical with the two equations of (3), the most general function of u and v which satisfies all of these equations cannot depend upon more than two linearly independent functions. It is impossible, therefore, that the four linearly independent functions $y', \cdots, y^{(4)}$ should satisfy an equation of form (11), i. e., it is impossible that the integrating surface S_y should degenerate into a curve.

The integrating surface of system (3) is not unique. However, the most general system of linearly independent solutions of (3), in the case considered, is of the form

$$\eta^{(i)} = \sum_{k=1}^{4} c_{ik} y^{(k)} \qquad (i = 1, 2, 3, 4),$$

where the determinant of the constant coefficients,

$$|c_{ik}|,$$

does not vanish. *The most general integrating surface of* (3) *is, therefore, a projective transformation of any particular one.* It is for this reason that we shall make use of (3) as a basis for the projective theory of surfaces.

If $aa' - 1$ vanishes, while not all of the conditions for an involutory system are satisfied, the system (3) has only three or fewer linearly independent solutions. Its integrating surface, therefore, degenerates into a plane or a straight line. For the purposes of the theory of surfaces this case may, therefore, be left aside.

It remains to examine the case when the equations of (3) are in involution. That case must also be excluded from our considerations. For we wish to construct a *projective* theory of surfaces; but, if the equations (3) are in involution, they have more than four linearly independent solutions in common, so that the most general integrating surface of (3) will not be merely a projective transformation of any particular one. The question which requires investigation is this: what kind of surfaces are thus excluded from consideration?

If we put in (3)

$$\bar{y} = \lambda y$$

where λ is an arbitrary function of u and v, and if we introduce new independent variables by putting

$$\bar{u} = \mu(u, v), \qquad \bar{v} = \nu(u, v),$$

another system of differential equations of form (1) will be obtained from (3) with $\bar{y}, \bar{u}, \bar{v}$, in place of y, u, v. Moreover, it is clear that, if the equations (3) are in involution, the same must be true of the transformed equations and conversely. For if they were not, they could have only four linearly independent solutions and the same would have to be true of the original system of equations.

Let equations (3) be in involution. They have an infinite number of linearly independent solutions. Let $y', \cdots, y^{(4)}$ be four of these. Transform system (3) by putting $\bar{y} = \lambda y$ where $\lambda = 1/y^{(4)}$. The new system of differential equations is again an involutory system and has the four linearly independent solutions

$$1, \qquad x = \frac{y'}{y^{(4)}}, \qquad y = \frac{y''}{y^{(4)}}, \qquad z = \frac{y^{(3)}}{y^{(4)}},$$

where x, y and z may be considered as non-homogeneous coördinates of the point of the integral surface whose homogeneous coördinates were $y', y'', y^{(3)}, y^{(4)}$. Moreover x, y, z are functions of u and v. Introduce x and y as independent variables. This is possible unless x should happen to be a function of y alone. In that case we might introduce x and z as independent variables. For otherwise x, y and z would be functions of a single variable $t = \phi(u, v)$, so that $y', \cdots, y^{(4)}$, as we have seen above, would satisfy one and the same linear partial differential equation of the first order. In that case there would be no integral surface, but an integral curve; for our purpose that case must be excluded. Accordingly two of the three functions x, y, z of u and v will be indepen-

dent, say x and y. We may then introduce x and y as independent variables in place of u and v. The transformed system of equations will be again an involutory system with the independent variables x and y, and of which $1, x, y$ and $z = f(x, y)$ are four linearly independent solutions. This system of equations will certainly be of form (1), and in most cases may at once be written in the form (3) by solving for θ_{xx} and θ_{yy}, denoting the dependent variable by θ. The system is then of the form

$$\theta_{xx} = a\theta_{xy} + b\theta_x + c\theta_y + d\theta,$$
$$\theta_{yy} = a'\theta_{xy} + b'\theta_x + c'\theta_y + d'\theta,$$

and it must be satisfied by $\theta = 1, x, y$ and $z = f(x, y)$, so that

$$b = c = d = b' = c' = d' = 0,$$

while $z = f(x, y)$ must satisfy the equations

$$\frac{\partial^2 z}{\partial x^2} = a\frac{\partial^2 z}{\partial x \partial y}, \qquad \frac{\partial^2 z}{\partial y^2} = a'\frac{\partial^2 z}{\partial x \partial y}, \tag{12}$$

which must, moreover, be in involution.

If, however, the system of form (1), which is obtained by the above process, cannot be solved for θ_{xx} and θ_{yy}, then as was shown in § 1, by a simple linear transformation of the independent variables x and y we may transform it into a system of form (3). This latter system must then have the solutions

$$1, \quad \alpha x + \beta y, \quad \gamma x + \delta y, \quad z, \quad \alpha\delta - \beta\gamma \neq 0,$$

where $\alpha, \beta, \gamma, \delta$ are constants, whence we conclude in the same way as before that z must satisfy an involutory system of form (12). But the equations (12) are in involution if, and only if,

$$aa' = 1, \qquad \frac{\partial a}{\partial y} + a\frac{\partial a}{\partial x} = 0, \tag{13}$$

so that

$$\frac{\partial^2 z}{\partial x^2}\frac{\partial^2 z}{\partial y^2} - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 = 0. \tag{14}$$

If x, y, z are cartesian coördinates it is well known that (14) is the condition for developable surfaces. But this condition is left invariant under projective transformations. Therefore, *the surfaces excluded from consideration are developables.* If the condition

$$\frac{\partial a}{\partial y} + a\frac{\partial a'}{\partial x} = 0 \tag{15}$$

were not satisfied, the surface would be a plane. All developables are thus

excluded. For let $z = f(x, y)$ be the equation of a developable. We may determine a from the equation

$$\frac{\partial^2 z}{\partial x^2} = a \frac{\partial^2 z}{\partial x \partial y}$$

if $\partial^2 z/\partial x \partial y \neq 0$. Since the surface is a developable, (14) must be satisfied, which gives

$$\frac{\partial^2 z}{\partial y^2} = \frac{1}{a} \frac{\partial^2 z}{\partial x \partial y};$$

i. e., z satisfies a system of form (12) for which $aa' = 1$. Moreover (15) must then also be satisfied unless z is a linear function of x and y, i. e., unless the surface is a plane. If, however,

$$\frac{\partial^2 z}{\partial x \partial y} = 0,$$

equation (14) shows that either $\partial^2 z/\partial x^2$ or $\partial^2 z/\partial y^2$ must vanish. Suppose we have

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial x^2} = 0.$$

This system is not of form (3), but may be put into that form by a linear transformation of the independent variables. It will be an involutory system as one may see even without making the transformation. For the most general solution of the above system is

$$z = c_1 x + \phi(y)$$

where $\phi(y)$ is an arbitrary function of y, and therefore contains an infinite number of arbitrary constants.

We may recapitulate as follows: 1. *Given a system of partial differential equations of form* (3), *whose integrability conditions are satisfied identically and for which* $aa' - 1$ *is not equal to zero. Its most general integrating surface is a projective transformation of any particular one; it does not degenerate into a curve and is not a developable.*

2. *If for such a system of equations, which is not involutory,* $aa' - 1$ *is equal to zero, its integrating surface degenerates into a plane or line.*

3. *If the system is involutory, its integral surfaces are either curves or developables and the most general integral curve or developable is not a mere projective transformation of any particular one.*

For the purpose of constructing a projective differential geometry of non-developable surfaces we may therefore confine ourselves to the first case. Moreover, *any* non-developable surface may be studied by means of such a system of equations. For let x, y, z be the non-homogeneous coördinates of a point on

any non-developable surface, and take x and y as independent variables. Then z will be a certain function of x and y, say

$$z = f(x, y).$$

We can determine two functions a and a' of x and y so that z will satisfy equations (12), provided that

$$\frac{\partial^2 z}{\partial x \partial y} \neq 0.$$

If, however,

$$\frac{\partial^2 z}{\partial x \partial y} = 0,$$

we may transform the independent variables by putting

$$\bar{x} = \alpha x + \beta y, \qquad \bar{y} = \gamma x + \delta y, \qquad \alpha\delta - \beta\gamma \neq 0,$$

where α, β, γ, δ are constants. These constants may always be chosen in such a way that $\partial^2 z/\partial \bar{x} \partial \bar{y}$ will not be zero, unless not only $\partial^2 z/\partial x \partial y$, but also $\partial^2 z/\partial x^2$ and $\partial^2 z/\partial y^2$ are equal to zero. In that case the surface would be a plane. Excepting this case it is, therefore, always possible to construct a system of form (12) for every surface. Since, moreover, the surface is not developable $aa' - 1$ will not vanish. If homogeneous coördinates be employed, and an arbitrary set of independent variables, a system of form (3) will be obtained for which $aa' - 1$ is different from zero, and which is not involutory.

Therefore, *the theory of a completely integrable non-involutory system of two partial differential equations of the second order, with two independent and one dependent variable, is identical with the projective differential geometry of non-degenerate, non-developable surfaces in three-dimensional space.*

§ 3. *General notions on invariants and covariants.*

Consider a system of form (3) whose integrability conditions are satisfied identically and for which $aa' - 1$ does not vanish. Let

$$y^{(k)} = f^{(k)}(u, v) \qquad (k = 1, 2, 3, 4), \tag{16}$$

be four linearly independent solutions of the system. Equations (16) determine an integral surface S_y of (3), and the most general integral surface of (3) will be a projective transformation of S_y.

But there are some, so to speak, accidental elements in the representation (16) of the surface S_y. In the first place, since the coördinates are homogeneous, multiplication of the four functions $y^{(k)}$ by an arbitrary function λ of u and v will not alter the surface. In the second place the independent variables may

be transformed arbitrarily. If, therefore, we make any transformation of the form

$$\bar{y} = \lambda(u, v)y, \qquad \bar{u} = \phi(u, v), \qquad \bar{v} = \psi(u, v), \tag{17}$$

$$\lambda \neq 0, \qquad \frac{\partial(\phi, \psi)}{\partial(u, v)} \neq 0,$$

the transformed system of differential equations will have the same integral surfaces as (3).

Those combinations of the coefficients of (3) which are unaltered in value by a transformation of form (17), will be called *invariants*. Such invariant functions as depend also upon y and its derivatives will be called *covariants*. Any projective property of a surface will be expressed by an invariant equation, or system of equations, i. e., by one which is left unchanged by transformations of the form (17). The covariants will determine other surfaces and other geometrical configurations which have a projective relation to the given one.

The determination of the invariants and covariants of system (3) under the transformations (17) constitutes, therefore, nothing more or less than a projective differential theory of surfaces. In this general form, however, it is exceedingly difficult. The calculations involved are so long and complicated as to become tedious and uninstructive. We have, therefore, decided to reduce (3) to a canonical form, a form which exists for every surface, so that there is no sacrifice in the generality of our considerations, in so far as they refer to any non-developable surface. It remains desirable, nevertheless, to construct directly an invariant theory for a system of form (3), or more generally still, for a system of form (1) which is not in the canonical form. This, however, is a question which we shall, for the present, leave untouched.

§ 4. *The intermediate form. The surface referred to its asymptotic curves.*

Transform the independent variables in (3) by putting

$$\bar{u} = \phi(u, v), \qquad \bar{v} = \psi(u, v).$$

Let $\bar{y}_u$, $\bar{y}_v$, etc., denote $\partial y/\partial \bar{u}$, $\partial y/\partial \bar{v}$, etc., and put

$$\phi_u = \frac{\partial \phi}{\partial u}, \qquad \phi_v = \frac{\partial \phi}{\partial v}, \qquad \phi_{uu} = \frac{\partial^2 \phi}{\partial u^2},$$

etc. Then we shall find

$$y_u = \phi_u \bar{y}_u + \psi_u \bar{y}_v, \qquad y_v = \phi_v \bar{y}_u + \psi_v \bar{y}_v, \tag{18}$$

and

$$\begin{aligned} y_{uu} &= \phi_u^2 \bar{y}_{uu} + 2\phi_u \psi_u \bar{y}_{uv} + \psi_u^2 \bar{y}_{vv} + \phi_{uu} \bar{y}_u + \psi_{uu} \bar{y}_v, \\ y_{uv} &= \phi_u \phi_v \bar{y}_{uu} + (\phi_u \psi_v + \psi_u \phi_v) \bar{y}_{uv} + \psi_u \psi_v \bar{y}_{vv} + \phi_{uv} \bar{y}_u + \psi_{uv} \bar{y}_v, \\ y_{vv} &= \phi_v^2 \bar{y}_{uu} + 2\phi_v \psi_v \bar{y}_{uv} + \psi_v^2 \bar{y}_{vv} + \phi_{vv} \bar{y}_u + \psi_{vv} \bar{y}_v. \end{aligned} \tag{19}$$

If these values be substituted into (3), if the resulting system of equations be solved for $\bar{y}_{uu}$ and $\bar{y}_{vv}$, and if the new coefficients be denoted by $\bar{a}, \bar{b}, \cdots, \bar{a}', \cdots, \bar{d}'$, we shall find

$$\bar{a} = \frac{a\psi_v^2 - 2\psi_u\psi_v + a'\psi_u^2}{\phi_u\psi_v + \phi_v\psi_u - a\phi_v\psi_v - a'\phi_u\psi_u},$$

(20)

$$\bar{a}' = \frac{a\phi_v^2 - 2\phi_u\phi_v + a'\phi_u^2}{\phi_u\psi_v + \phi_v\psi_u - a\phi_v\psi_v - a'\phi_u\psi_u},$$

the other coefficients being fractions with the common denominator

$$\Delta = (\phi_u\psi_v - \phi_v\psi_u)(\phi_u\psi_v + \phi_v\psi_u - a\phi_v\psi_v - a'\phi_u\psi_u), \tag{21}$$

the first factor of which will not be equal to zero since ϕ and ψ are independent functions of u and v.

Let $\phi_v : \phi_u$ and $\psi_v : \psi_u$ be the two roots of the quadratic

$$at^2 - 2t + a' = 0,$$

so that, for example,

$$\frac{\phi_v}{\phi_u} = \frac{1 + \sqrt{1 - aa'}}{a}, \qquad \frac{\psi_v}{\psi_u} = \frac{1 - \sqrt{1 - aa'}}{a},$$

or

$$(1 + \sqrt{1 - aa'})\phi_u - a\phi_v = 0, \qquad (1 - \sqrt{1 - aa'})\psi_u - a\psi_v = 0. \tag{22}$$

These equations are distinct since $1 - aa'$ does not vanish, so that the Jacobian of the functions ϕ and ψ will not vanish as a consequence of (22). Assume $a \neq 0$. Then neither ϕ_u nor ψ_u can vanish since, as a consequence of (22), ϕ_v or ψ_v would then also vanish, and this would reduce the Jacobian of ϕ and ψ to zero. The denominator of the expressions (20), or the second factor of Δ, becomes

$$2\frac{\phi_u\psi_u(1 - aa')}{a},$$

which is different from zero. The transformation indicated, therefore, makes $\bar{a} = \bar{a}' = 0$. We have proved this under the assumption that $a \neq 0$. If $a = 0$ and $a' \neq 0$ we may still make $\bar{a} = \bar{a}' = 0$ by a very slight modification of the above argument. If $a = a' = 0$ the original system has the required form without transformation.

Suppose this to be the case. Equations (20) show that this form will not be altered if we make either the transformation

$$\bar{u} = \phi(u), \qquad \bar{v} = \psi(v),$$

or

$$\bar{u} = \phi(v), \qquad \bar{v} = \psi(u),$$

where ϕ and ψ are functions of a single argument.

We may recapitulate as follows. *A system of form* (3), *whose integrating*

surfaces are not developables, may always be reduced to the form

$$\begin{aligned} y_{uu} &= by_u + cy_v + dy, \\ y_{vv} &= b'y_u + c'y_v + d'y, \end{aligned} \tag{23}$$

which shall be called its intermediate form. The reduction requires the integration of the partial differential equation

$$a'\left(\frac{\partial \chi}{\partial u}\right)^2 - 2\frac{\partial \chi}{\partial u}\frac{\partial \chi}{\partial v} + a\left(\frac{\partial \chi}{\partial v}\right)^2 = 0. \tag{24}$$

If the independent variables have been selected in such a way as to cause the system to assume this intermediate form, the most general transformation of the independent variables which does not disturb this form is either

$$\bar{u} = \phi(u), \qquad \bar{v} = \psi(v), \tag{25}$$

or

$$\bar{u} = \phi(v), \qquad \bar{v} = \psi(u), \tag{26}$$

where ϕ and ψ are arbitrary functions of their arguments.

It is easy to find the geometrical significance of this intermediate form. Consider a curve $v = \text{const.}$ on the surface S_y. Let λ and μ be arbitrary functions of u and v. Then the four values of $\lambda y^{(k)} + \mu y_u^{(k)}$ will be the coördinates of an arbitrary point on the tangent of this curve constructed at P_y; similarly $\lambda y^{(k)} + \mu y_u^{(k)} + \nu y_{uu}^{(k)}$ will represent the coördinates of an arbitrary point of the osculating plane of the same curve at P_y. An aggregate of the form $\rho y^{(k)} + \sigma y_u^{(k)} + \tau y_v^{(k)}$ represents a point in the plane tangent to the surface S_y at P_y. Equations (23) show that any arbitrary point of the plane which osculates the curve $v = \text{const.}$ at P_y, is a point of the plane which is tangent to the surface at P_y. The curve $v = \text{const.}$, therefore, has the property that its osculating planes are at the same time tangent planes of the surface. Such curves are called *asymptotic* curves. From our considerations one easily deduces that there are just two families of ∞^1 asymptotic curves upon the surface. They are the curves $u = \text{const.}$ and $v = \text{const.}$ determined by the above process.

Reduction of a system of form (3) to its intermediate form is, therefore, equivalent to the determination of the asymptotic curves of its integrating surfaces.

It is also clear why the transformations (25) or (26) leave the intermediate form unchanged. They merely transform the asymptotic curves of each family into themselves, or else interchange the two families.

§ 5. *The integrability conditions for a system in the intermediate form. The canonical form.*

Consider a system of the intermediate form

$$\begin{aligned} y_{uu} + 2ay_u + 2by_v + cy &= 0, \\ y_{vv} + 2a'y_u + 2b'y_v + c'y &= 0. \end{aligned} \tag{27}$$

We proceed to develop its integrability conditions explicitly. By differentiation, we find from (27),

$$
\begin{aligned}
y_{uuu} &= p_1 y_{uv} + p_2 y_u + p_3 y_v + p_4 y, \\
y_{uuv} &= q_1 y_{uv} + q_2 y_u + q_3 y_v + q_4 y, \\
y_{uvv} &= r_1 y_{uv} + r_2 y_u + r_3 y_v + r_4 y, \\
y_{vvv} &= s_1 y_{uv} + s_2 y_u + s_3 y_v + s_4 y,
\end{aligned} \tag{28}
$$

where

$$
\begin{aligned}
&p_1 = -2b, \quad && p_2 = 4a^2 - 2a_u - c, && p_3 = 4ab - 2b_u, && p_4 = 2ac - c_u, \\
&q_1 = -2a, && q_2 = 4a'b - 2a_v, && q_3 = 4bb' - 2b_v - c, && q_4 = 2bc' - c_v, \\
&r_1 = -2b', && r_2 = 4aa' - 2a'_u - c', && r_3 = 4a'b - 2b'_u, && r_4 = 2a'c - c'_u, \\
&s_1 = -2a', && s_2 = 4a'b' - 2a'_v, && s_3 = 4b'^2 - 2b'_v - c', && s_4 = 2b'c' - c'_v.
\end{aligned} \tag{29}
$$

The integrability conditions are obtained by writing

$$
\frac{\partial y_{uuu}}{\partial v} = \frac{\partial y_{uuv}}{\partial u}, \qquad \frac{\partial y_{uuv}}{\partial v} = \frac{\partial y_{uvv}}{\partial u}, \qquad \frac{\partial y_{uvv}}{\partial v} = \frac{\partial y_{vvv}}{\partial u}.
$$

It will be found that the first and last of these three conditions are fulfilled identically. We find further

$$
\begin{aligned}
\frac{\partial y_{uuv}}{\partial v} &= \left(\frac{\partial q_1}{\partial v} + q_1 r_1 + q_2\right) y_{uv} + \left(\frac{\partial q_2}{\partial v} + q_1 r_2 - 2a' q_3\right) y_u \\
&\quad + \left(\frac{\partial q_3}{\partial v} + q_1 r_3 - 2b' q_3 + q_4\right) y_v + \left(\frac{\partial q_4}{\partial v} + q_1 r_4 - c' q_3\right) y, \\
\frac{\partial y_{uvv}}{\partial u} &= \left(\frac{\partial r_1}{\partial u} + r_1 q_1 + r_3\right) y_{uv} + \left(\frac{\partial r_2}{\partial u} + r_1 q_2 - 2a r_2 + r_4\right) y_u \\
&\quad + \left(\frac{\partial r_3}{\partial u} + r_1 q_3 - 2b r_2\right) y_v + \left(\frac{\partial r_4}{\partial u} + r_1 q_4 - c r_2\right) y.
\end{aligned}
$$

In order that these two expressions may be equal for all values of y, y_u, y_v, y_{uv}, the corresponding coefficients in the two expressions must be identical. We thus find the following four conditions:

$$
\begin{aligned}
&a_v - b'_u = 0, \\
&a'_{uu} + c'_u - 2a' a_u - 2aa'_u - (a_{vv} + 2b' a_v - 2ba'_v - 4a' b_v) = 0, \\
&b_{vv} + c_v - 2bb'_v - 2b' b_v - (b'_{uu} + 2ab'_u - 2a' b_u - 4ba'_u) = 0, \\
&c'_{uu} - 4ca'_u - 2a' c_u + 2ac'_u - (c_{vv} - 4c' b_v - 2bc'_v + 2b' c_v) = 0.
\end{aligned} \tag{30}
$$

The conditions (30) *must be satisfied in order that the general solution of* (27) *may depend upon just four arbitrary constants, for this is the only case of*

interest from the point of view of the theory of surfaces, as was explained in § 2. These integrability conditions (30) are symmetrically constructed from the coefficients of (27) in such a way that the operation which consists in interchanging simultaneously the letters in the two rows of the symbol

$$\begin{pmatrix} a & a' & c & u \\ b' & b & c' & v \end{pmatrix},$$

converts the first and fourth equations into themselves and interchanges the other two.

From the significance of the conditions (30), it is clear that they must form an invariant system of equations. In other words, if the variables in (27) are changed arbitrarily, but in such a way as not to disturb the form of the system, the integrability conditions for the transformed system of equations will be fulfilled if they are fulfilled for the original system, and conversely.

On account of the first condition of (30) we may write

$$a = p_u, \qquad b' = p_v. \tag{31}$$

Transform (27) by putting

$$y = \lambda \bar{y},$$

where λ is a function of u and v. We find a new system of the same form for y whose coefficients are

$$\bar{a} = a + \frac{\lambda_u}{\lambda}, \qquad \bar{b} = b, \qquad \bar{c} = c + \frac{\lambda_{uu} + 2a\lambda_u + 2b\lambda_v}{\lambda},$$

$$\bar{a}' = a', \qquad \bar{b}' = b' + \frac{\lambda_v}{\lambda}, \qquad \bar{c}' = c' + \frac{\lambda_{vv} + 2a'\lambda_u + 2b'\lambda_v}{\lambda}.$$

Owing to (31) we may make $\bar{a} = \bar{b}' = 0$ by putting

$$\lambda = e^{-p},$$

so that

$$\frac{\lambda_u}{\lambda} = -p_u, \qquad \frac{\lambda_v}{\lambda} = -p_v,$$

$$\frac{\lambda_{uu}}{\lambda} = -p_{uu} + p_u^2, \qquad \frac{\lambda_{vv}}{\lambda} = -p_{vv} + p_v^2,$$

whence

$$\bar{c} = c - a_u - a^2 - 2bb' = f,$$
$$\bar{c}' = c' - b'_v - b'^2 - 2aa' = g. \tag{32}$$

It becomes possible, therefore, to put the system (27) *into the form*

$$y_{uu} + 2by_v + fy = 0,$$
$$y_{vv} + 2a'y_u + gy = 0, \tag{33}$$

which is characterized by the conditions $a = b' = 0$. We shall speak of (33) as the *canonical form* of the given system of differential equations. It may be obtained from the intermediate form in the way just indicated. *The first of the four integrability conditions* (30) *is satisfied identically. The other three assume the simplified form:*

$$\begin{aligned} a'_{uu} + g_u + 2ba'_v + 4a'b_v &= 0, \\ b_{vv} + f_v + 2a'b_u + 4ba'_u &= 0, \\ g_{uu} - f_{vv} - 4ga'_u - 2a'f_u + 4fb_v + 2bg_v &= 0. \end{aligned} \tag{34}$$

§ 6. *Seminvariants and semi-covariants.*

Consider again the system (27) of differential equations, and let it be transformed by putting

$$y = \lambda \bar{y}, \tag{35}$$

where λ is an arbitrary function of u and v. Those functions of the coefficients of (27), of y, and of the derivatives of these quantities, which are left unchanged by every such transformation, shall be called *seminvariants* or *semi-covariants* according as they do not, or do contain y and its derivatives.

We find from (35)

$$\begin{aligned} y_u &= \lambda \bar{y}_u + \lambda_u \bar{y}, \qquad y_v = \lambda \bar{y}_v + \lambda_v \bar{y}, \\ y_{uu} &= \lambda \bar{y}_{uu} + 2\lambda_u \bar{y}_u + \lambda_{uu} \bar{y}, \\ y_{uv} &= \lambda \bar{y}_{uv} + \lambda_u \bar{y}_v + \lambda_v \bar{y}_u + \lambda_{uv} \bar{y}, \\ y_{vv} &= \lambda \bar{y}_{vv} + 2\lambda_v \bar{y}_v + \lambda_{vv} \bar{y}, \end{aligned} \tag{36}$$

which gives, upon substitution into (27),

$$\begin{aligned} \bar{y}_{uu} + 2\bar{a}\bar{y}_u + 2\bar{b}\bar{y}_v + \bar{c}\bar{y} &= 0, \\ \bar{y}_{vv} + 2\bar{a}'\bar{y}_u + 2\bar{b}'\bar{y}_v + \bar{c}'\bar{y} &= 0, \end{aligned}$$

where

$$\begin{aligned} \bar{a} &= a + \frac{\lambda_u}{\lambda}, \qquad \bar{b} = b, \qquad \bar{c} = c + \frac{\lambda_{uu} + 2a\lambda_u + 2b\lambda_v}{\lambda}, \\ \bar{a}' &= a', \qquad \bar{b}' = b' + \frac{\lambda_v}{\lambda}, \qquad \bar{c}' = c' + \frac{\lambda_{vv} + 2a'\lambda_u + 2b'\lambda_v}{\lambda}. \end{aligned} \tag{37}$$

We find the following seminvariants

$$\begin{aligned} &a', \qquad b, \\ &f = c - a_u - a^2 - 2bb', \qquad g = c' - b'_v - b'^2 - 2aa'. \end{aligned} \tag{38}$$

The derivatives of these four quantities are also seminvariants. *Every semin-*

variant is a function of a', b, f, g and of their derivatives. That this is true may be, most simply, seen as follows: A seminvariant has the same value for system (27) as for any system obtained from it by a transformation of form (35). Make the transformation of form (35), which reduces (27) to its canonical form according to the method of § 5. The resulting system (33) has precisely a', b, f and g as its coefficients. Every seminvariant can, therefore, contain only these quantities and their derivatives, as was to be shown.

Semi-covariants contain y, y_u, y_v, y_{uv}, etc., in addition to the coefficients of (27). Since all higher derivatives of y may be expressed in terms of the four quantities

$$y, y_u, y_v, y_{uv},$$

we may confine our search for semi-covariants to such as involve them. We find, from (36),

$$y = \frac{1}{\lambda}\bar{y},$$

$$\bar{y}_u = \frac{1}{\lambda}\left(y_u - \frac{\lambda_u}{\lambda}y\right), \qquad \bar{y}_v = \frac{1}{\lambda}\left(y_v - \frac{\lambda_v}{\lambda}y\right), \tag{39}$$

$$\bar{y}_{uv} = \frac{1}{\lambda}\left[y_{uv} - \frac{\lambda_u}{\lambda}y_v - \frac{\lambda_v}{\lambda}y_u + \left(\frac{2\lambda_u\lambda_v}{\lambda^2} - \frac{\lambda_{uv}}{\lambda}\right)y\right],$$

so that y itself is a *relative* semi-covariant. By making use of (37) we find three other relative semi-covariants

$$z = y_u + ay, \qquad \rho = y_v + b'y,$$

$$\sigma = y_{uv} + b'y_u + ay_v + \tfrac{1}{2}(a_v + b'_u + 2ab')y; \tag{40}$$

in fact these quantities satisfy the equations

$$\bar{y} = \lambda y, \qquad \bar{z} = \lambda z, \qquad \bar{\rho} = \lambda\rho, \qquad \bar{\sigma} = \lambda\sigma,$$

so that their ratios are absolute semi-covariants. The four semi-covariants y, z, ρ, σ are clearly independent. Moreover equations (40) show that y, y_u, y_v, y_{uv} may be expressed in terms of y, z, ρ, σ. Since every relative semicovariant can be expressed in terms of y, y_u, y_v, y_{uv}, it may be also expressed in terms of y, z, ρ, σ. By reducing to the canonical form it becomes obvious that *all semi-covariants can be expressed as functions of y, z, ρ, σ and of seminvariants.*

§ 7. *Invariants and covariants.*

In order to find invariants we introduce transformations of the independent variables which leave the intermediate form of the system of partial differential equations unaltered. These are either of form (25) or (26). Since (26) may be obtained from (25) by the simple transformation

$$u = \bar{v}, \qquad v = \bar{u},$$

which merely interchanges the two families of asymptotic lines, it becomes an easy matter to find the invariants under the general transformation if those under transformation (25) have been found. It is desirable, moreover, to keep the two families of asymptotic curves separate. For these reasons we confine ourselves to the transformation (25) or as we shall now write it

$$\bar{u} = \alpha(u), \qquad \bar{v} = \beta(v), \tag{41}$$

denoting the new independent variables by $\bar{u}$ and $\bar{v}$ where α is a function of u alone, and β is a function of the single variable v. Upon making the transformation (41), we find

$$\begin{aligned} y_u &= \alpha_u \frac{\partial y}{\partial \bar{u}}, \qquad y_{uu} = \alpha_u^2 \frac{\partial^2 y}{\partial \bar{u}^2} + \alpha_{uu} \frac{\partial y}{\partial \bar{u}}, \\ y_v &= \beta_v \frac{\partial y}{\partial \bar{v}}, \qquad y_{vv} = \beta_v^2 \frac{\partial^2 y}{\partial \bar{v}^2} + \beta_{vv} \frac{\partial y}{\partial \bar{v}}, \\ y_{uv} &= \alpha_u \beta_v \frac{\partial^2 y}{\partial \bar{u}\, \partial \bar{v}}. \end{aligned} \tag{42}$$

Equations (27) are transformed into a similar system, whose coefficients are

$$\begin{aligned} \bar{a} &= \frac{1}{\alpha_u}(a + \tfrac{1}{2}\eta), \qquad \bar{b} = \frac{\beta_v b}{\alpha_u^2}, \qquad \bar{c} = \frac{c}{\alpha_u^2}, \\ \bar{a}' &= \frac{\alpha_u a'}{\beta_v^2}, \qquad \bar{b}' = \frac{1}{\beta_v}(b' + \tfrac{1}{2}\zeta), \qquad \bar{c}' = \frac{c'}{\beta_v^2}, \end{aligned} \tag{43}$$

where

$$\eta = \frac{\alpha_{uu}}{\alpha_u}, \qquad \zeta = \frac{\beta_{vv}}{\beta_v}. \tag{44}$$

These equations show that a' and b are (relative) invariants as well as seminvariants.

Since invariants are also seminvariants, they can only be functions of a', b, f, g and of their derivatives. It becomes necessary, therefore, to investigate how the transformation (41) affects these seminvariants. Invariants are such functions of seminvariants as are left unaltered by this transformation.

We find, from (43),

$$\begin{aligned} \frac{\partial \bar{a}}{\partial \bar{u}} &= \frac{1}{\alpha_u^2}[a_u - \eta a + \tfrac{1}{2}(\eta_u - \eta^2)], & \frac{\partial \bar{a}}{\partial \bar{v}} &= \frac{a_v}{\alpha_u \beta_v}, \\ \frac{\partial \bar{b}'}{\partial \bar{u}} &= \frac{b'_u}{\alpha_u \beta_v}, & \frac{\partial \bar{b}'}{\partial \bar{v}} &= \frac{1}{\beta_v^2}[b'_v - \zeta b' + \tfrac{1}{2}(\zeta_v - \zeta^2)], \\ \frac{\partial \bar{a}'}{\partial \bar{u}} &= \frac{1}{\beta_v^2}(a'_u + \eta a'), & \frac{\partial \bar{a}'}{\partial \bar{v}} &= \frac{\alpha_u}{\beta_v^3}(a'_v - 2\zeta a'), \\ \frac{\partial \bar{b}}{\partial \bar{u}} &= \frac{\beta_v}{\alpha_u^3}(b_u - 2\eta b), & \frac{\partial \bar{b}}{\partial \bar{v}} &= \frac{1}{\alpha_u^2}(b_v + \zeta b). \end{aligned} \tag{45}$$

From the expressions (38) for f and g, and the above equations, we find

$$\bar{f} = \frac{1}{\alpha_u^2}(f - b\zeta - \tfrac{1}{2}\mu), \qquad \bar{g} = \frac{1}{\beta_v^2}(g - a'\eta - \tfrac{1}{2}\nu), \tag{46}$$

where

$$\mu = \eta_u - \tfrac{1}{2}\eta^2, \qquad \nu = \zeta_v - \tfrac{1}{2}\zeta^2. \tag{47}$$

We have further

$$\frac{\partial^2 \bar{a}'}{\partial \bar{u}^2} = \frac{1}{\alpha_u \beta_v^2}[a'_{uu} + \eta a'_u + \eta_u a'],$$

$$\frac{\partial^2 \bar{a}'}{\partial \bar{u}\,\partial \bar{v}} = \frac{1}{\beta_v^3}[a'_{uv} + \eta a'_v - 2\zeta a'_u - 2\eta\zeta a'], \tag{48a}$$

$$\frac{\partial^2 \bar{a}'}{\partial \bar{v}^2} = \frac{\alpha_u}{\beta_v^4}[a'_{vv} - 5\zeta a'_v - 2\zeta_v a' + 6\zeta^2 a'],$$

and

$$\frac{\partial^2 \bar{b}}{\partial \bar{u}^2} = \frac{\beta_v}{\alpha_u^4}[b_{uu} - 5\eta b_u - 2\eta_u b + 6\eta^2 b],$$

$$\frac{\partial^2 \bar{b}}{\partial \bar{u}\,\partial \bar{v}} = \frac{1}{\alpha_u^3}[b_{uv} + \zeta b_u - 2\eta b_v - 2\eta\zeta b], \tag{48b}$$

$$\frac{\partial^2 \bar{b}}{\partial \bar{v}^2} = \frac{1}{\alpha_u^2 \beta_v}[b_{vv} + \zeta b_v + \zeta_v b].$$

By means of these equations it is easy to verify that

$$\bar{h} = \frac{\beta_v^2 h}{\alpha_u^6}, \qquad \bar{k} = \frac{\alpha_u^2 k}{\beta_v^6}, \tag{49}$$

where

$$h = b^2(f + b_v) - \tfrac{1}{4}bb_{uu} + \tfrac{5}{16}b_u^2, \qquad k = a'^2(g + a'_u) - \tfrac{1}{4}a'a'_{vv} + \tfrac{5}{16}a'^2_v, \tag{50}$$

so that we have found four invariants a', b, h and k. We shall show how other invariants may be derived from these four by certain differentiation processes. Put

$$A = a'b^2, \qquad B = a'^2 b, \qquad H = a'h, \qquad K = bk \tag{51}$$

so that

$$\bar{A} = \frac{A}{\alpha_u^3}, \qquad \bar{B} = \frac{B}{\beta_v^3}, \qquad \bar{H} = \frac{H}{\alpha_u^5}, \qquad \bar{K} = \frac{K}{\beta_v^5}, \tag{52}$$

it being assumed that a' and b are different from zero.

Introduce two symbols of operation

$$U = a'\frac{\partial}{\partial u}, \qquad V = b\frac{\partial}{\partial v}. \tag{53}$$

Then clearly

$$U(B), \qquad V(A), \qquad U(K), \qquad V(H) \tag{54}$$

are also invariants. In fact

$$(55)\quad \overline{V}(\overline{A})=\frac{V(A)}{\alpha_u^5},\quad \overline{U}(\overline{B})=\frac{U(B)}{\beta_v^5},\quad \overline{V}(\overline{H})=\frac{V(H)}{\alpha_u^7},\quad \overline{U}(\overline{K})=\frac{U(K)}{\beta_v^7},$$

so that it becomes clear that every repetition of these two processes gives rise to an invariant.

Further, from two invariants λ and μ for which

$$\bar{\lambda}=\frac{1}{\alpha_u^l}\lambda,\qquad \mu=\frac{1}{\alpha_u^m}\mu,$$

we can always form another, its Wronskian. In fact we find

$$\overline{\lambda^m\mu^{-l}}=\lambda^m\mu^{-l},$$

whence, by logarithmic differentiation,

$$\overline{m\frac{\lambda_u}{\lambda}-l\frac{\mu_u}{\mu}}=\frac{1}{\alpha_u}\left(m\frac{\lambda_u}{\lambda}-l\frac{\mu_u}{\mu}\right),$$

so that the function

$$(56)\qquad (\mu,\lambda_u)=m\mu\lambda_u-l\lambda\mu_u$$

is an invariant for which

$$(57)\qquad \overline{(\mu\lambda_u)}=\frac{(\mu\lambda_u)}{\alpha_u^{l+m+1}}.$$

We shall speak of $(\mu\lambda_u)$ as the Wronskian of μ and λ with respect to u.

By combining the Wronskian process with the operations U and V, it clearly becomes possible to deduce an infinity of invariants from A, B, H and K. We proceed to show that any invariant is a function of those obtained in this manner.

In the first place, the equations which define A, B, H and K, show that a', b, f and g may be expressed in terms of A, B, H, K and of their derivatives. Since all invariants are functions of a', b, f, g, and of their derivatives, any invariant may be expressed as a function of A, B, H, K and of the derivatives of these quantities.

Introduce infinitesimal transformations by putting

$$(58)\qquad \alpha(u)=u+\phi(u)\delta t\qquad \beta(v)=v+\psi(v)\delta t,$$

where δt is an infinitesimal. Then

$$(59)\qquad \begin{aligned}&\alpha_u=1+\phi_u\delta t,\qquad &&\alpha_v=0,\\ &\beta_u=0,\qquad &&\beta_v=1+\psi_v\delta t,\\ &\alpha_u\beta_v-\alpha_v\beta_u=1+(\phi_u+\psi_v)\delta t.\end{aligned}$$

The infinitesimal transformations of A, B, H and K become

$$\delta A = -3\phi_u A\delta t,\ \delta B = -3\psi_v B\delta t,\ \delta H = -5\phi_u H\delta t,\ \delta K = -5\psi_v K\delta t. \tag{6}$$

The infinitesimal transformation, δF, of any function of A, B, etc., being known, we may easily find the infinitesimal transformations δF_u and δF_v of its first derivatives. In fact we shall have

$$\begin{aligned}\delta F_u &= \frac{\partial \bar{F}}{\partial \bar{u}} - \frac{\partial F}{\partial u} = \frac{\partial (F+\delta F)}{\partial u}\frac{1}{1+\phi_u\delta t} - \frac{\partial F}{\partial u} = \frac{\partial(\delta F)}{\partial u} - \phi_u F_u\delta t,\\ \delta F_v &= \frac{\partial \bar{F}}{\partial \bar{v}} - \frac{\partial F}{\partial v} = \frac{\partial (F+\delta F)}{\partial v}\frac{1}{1+\psi_v\delta t} - \frac{\partial F}{\partial v} = \frac{\partial(\delta F)}{\partial v} - \psi_v F_v\delta t.\end{aligned} \tag{61}$$

Thus we find

$$\begin{aligned}\delta A_u &= -(3\phi_{uu}A + 4\phi_u A_u)\delta t, & \delta A_v &= -(3\phi_u + \psi_v)A_v\delta t,\\ \delta B_u &= -(\phi_u + 3\psi_v)B_u\delta t, & \delta B_v &= -(3\psi_{vv}B + 4\psi_v B_v)\delta t,\\ \delta H_u &= -(5\phi_{uu}H + 6\phi_u H_u)\delta t, & \delta H_v &= -(5\phi_u + \psi_v)H_v\delta t,\\ \delta K_u &= -(\phi_u + 5\psi_v)K_u\delta t, & \delta K_v &= -(5\psi_{vv}K + 6\psi_v K_v)\delta t.\end{aligned} \tag{62}$$

All absolute invariants θ which involve only A, B, H, K and their first derivatives must, therefore, satisfy the system of partial differential equations obtained by forming the infinitesimal transformation $\delta\theta$ of an arbitrary function of these arguments and equating to zero the coefficients of ϕ_{uu}, ψ_{vv}, ϕ_u and ψ_v. If we are looking for relative invariants we may dispense with the latter two equations, as the following consideration will show. Consider the special transformation for which

$$\alpha(u) = cu, \qquad \beta(v) = c'v, \tag{63}$$

so that

$$\alpha_u = c, \qquad \beta_v = c',$$

where c and c' are arbitrary constants. Then

$$\bar{A} = c^{-3}A, \qquad \bar{B} = (c')^{-3}B, \quad \text{etc.},$$

$$\bar{A}_u = c^{-4}A_u, \quad \text{etc.}$$

In general, let P be a combination of the quantities A, B, H, K and of their derivatives, and suppose that the effect of the transformation (63) is to convert it into $\bar{P}$, where

$$\bar{P} = c^{-p}c'^{-p'}P.$$

We shall then say that P has an α-weight equal to p and a β-weight equal to p'. Thus the weights of A and B are $(+3, 0)$ and $(0, +3)$ respectively. It is clear that differentiation with respect to u and v diminishes the α- or β-

weight respectively by unity. The weights of a product are the sums of the respective weights of the factors.

We shall say of a sum that it is *isobaric* of weights w, w' if each of its terms has the weights w, w'. It is clear that every absolute invariant is isobaric of weights 0, 0. *Every relative invariant must be isobaric.*

We obtain an infinitesimal transformation of the form (63) if we let ϕ_u and ψ_v in (60) and (62) represent constants. Under this assumption, if F is any isobaric function of weights w and w', the infinitesimal transformation of F will be of the form

$$\delta F = -(w\phi_u + w'\psi_v)F\delta t. \tag{64}$$

If ϕ_u and ψ_v are not constants, the expression for δF will contain other terms which will have ϕ_{uu}, ψ_{vv}, etc. as factors. If, however F is an irreducible (relative or absolute) invariant, the equation $\delta F = 0$ must be a consequence of $F = 0$, which implies that if F is an invariant of weights w, w', the equation (64) must be satisfied even if ϕ_u and ψ_v are not constants.*

Suppose then that we form the general expression for δF, where F depends upon A, B, H, K and the derivatives of these quantities up to any order n, and equate to zero the coefficients of all of the derivatives of the arbitrary functions ϕ and ψ, including ϕ_u and ψ_v. We thus obtain a complete system S of partial differential equations, whose solutions are the *absolute* invariants up to the nth order. If we omit the two equations of this system obtained by equating to zero the coefficients of ϕ_u and ψ_v, we obtain a system S' of partial differential equations satisfied by all relative invariants. Moreover, only such solutions of this system S' will be relative invariants for which (64) is satisfied, i. e., only isobaric solutions of the system are admissible as invariants. The relative invariants are certainly not all of weight zero. We know in fact that A, B, H and K are independent relative invariants, both of whose weights are not zero. Any relative invariant can, however, by multiplication with properly chosen powers of A and B be converted into an absolute invariant. Consequently there are two more independent relative invariants than absolute invariants. If the system S' were not a complete system, the difference between the number of relative and absolute invariants could not be equal to two. *The system S' is, therefore, a complete system and every isobaric solution of S' is a relative invariant.*

All relative invariants which involve only A, B, H, K and their first derivatives are, therefore, isobaric solutions of the complete system of two equations

$$3A\frac{\partial\theta}{\partial A_u} + 5H\frac{\partial\theta}{\partial H_u} = 0, \qquad 3B\frac{\partial\theta}{\partial B_v} + 5K\frac{\partial\theta}{\partial K_v} = 0,$$

* For the coefficients of ϕ_{uu}, ψ_{vv}, etc. cannot vanish in consequence of $F = 0$, if we assume F to be irreducible, since their coefficients are of weights lower than w, w'. The complete discussion would be the same as that in the author's treatise *Projective Differential Geometry*, etc., p. 22.

which involves (in part with zero coefficients) twelve independent variables. It has therefore ten independent solutions. These may be selected in the form

$$A,\ B,\ H,\ K,\ V(A),\ U(B),\ V(H),\ U(K), \tag{65a}$$

together with the two Wronskians

$$(A, H_u), \qquad (B, K_v). \tag{65b}$$

For they are clearly independent. In fact, taken in this order, each of these functions involves one variable which does not occur in any of the preceding ones.

All relative invariants which involve also the second derivatives of A, B, H, K will be isobaric solutions of a complete system of four independent equations with twenty-four independent variables. They are, therefore, twenty in number. Ten of these we have already found. The other ten may be taken in the form

$$\begin{aligned} &V^2(A),\ V^2(H),\ V(A, H_u),\ \big(A, V(A)_u\big),\ \big(A, (A, H_u)_u\big), \\ &U^2(B),\ U^2(K),\ U(B, K_v),\ \big(B, U(B)_v\big),\ \big(B, (B, K_v)_v\big); \end{aligned} \tag{66}$$

for it is apparent that these are independent.

We have now determined all invariants up to and including those of the second order; they are isobaric functions of the twenty invariants (65) and (66). We make the following remarks. Let an invariant be called an α- or a β-invariant if only one of its two weights is different from zero, so that the β-weight of an α-invariant is zero, etc. Our invariants have been so chosen that half of them are α- and half of them β-invariants. Consider the five α-invariants of the second order. They are those in the first row of (66). Two of them, the first and the fourth do not contain H, and contain only two of the three derivatives of the second order of A, viz., A_{vv} and A_{uv}. The other three α-invariants contain each a different one of the three second derivatives of H. A similar classification may be made of the β-invariants. Suppose we had determined all of the invariants up to and including those of the $n-1$th order. To determine those of the nth order we should have to solve a system of partial differential equations containing two more equations and $4(n+1)$ more variables than the system for the invariants of order $n-1$. There will be, therefore, $4n+2$ invariants which involve the nth derivatives of A, B, H, K. Similarly there will be just $4n-2$ invariants of the $n-1$th order.

Let us suppose that what we have found to be true for the invariants of the second order is also true for those of order $n-1$, i. e., let us suppose that $2n-1$ of them are α- and $2n-1$ of them β-invariants. Denote these α-invariants of the $n-1$th order by

$$I_1,\ I_2,\ \cdots,\ I_{n-1}; \qquad I'_1,\ I'_2,\ \cdots,\ I'_n.$$

The notation indicates a division of these α-invariants into two classes; those of the first class do not depend upon H and its derivatives, those of the second class do. Let

$$P^{(n-1)}_{u_r, v_s} \qquad (r+s=n-1),$$

denote the derivative of P taken r times with respect to u and s times with respect to v. Then we shall assume that $I_1, I_2, \cdots, I_{n-1}$ contain in this order $A^{(n-1)}_{u_0, v_{n-1}}, A^{(n-1)}_{u_1, v_{n-2}}, \cdots, A^{(n-1)}_{u_{n-2}, v_1}$, and that no function I_k contains $A^{(n-1)}_{u_{n-1}, v_0}$.

We assume further that the functions $I'_1, I'_2, \cdots, I'_n$ taken in this order contain respectively $H^{(n-1)}_{u_0, v_{n-1}}, H^{(n-1)}_{u_1, v_{n-2}}, \cdots, H^{(n-1)}_{u_{n-1}, v_0}$.

The $2n-1$ functions

$$V(I_k), \qquad V(I'_k) \tag{67}$$

will then be α-invariants of the nth order. Those of the first set contain all of the derivatives of A of the nth order except $A^{(n)}_{u_{n-1}, v_1}$, $A^{(n)}_{u_n, v_0}$. Those of the second set contain all of the derivatives of H of the nth order except $H^{(n)}_{u_n, v_0}$.

Consider the Wronskian of A and I_{n-1} with respect to u. It will be an α-invariant which will contain the derivative $A^{(n)}_{u_{n-1}, v_1}$. Similarly the Wronskian of A and I'_n will be an α-invariant which contains $H^{(n)}_{u_n, v_0}$. If we add these two α-invariants to the $2n-1$ of (67), we shall have $2n+1$ independent α-invariants of order n which can be divided into two classes precisely in the same way as was assumed to be the case for those of the $n-1$th order. Since this classification was actually possible for the invariants of 2, it follows that this same classification will actually hold for any order n.

We have seen, therefore, that the Wronskian process and the process V enables us to obtain $2n+1$ independent α-invariants of order n from A and H. Similarly, by means of the Wronskian process and the process U we may obtain $2n+1$ independent β-invariants of order n from B and K. It only remains to show that these $4n+2$ invariants are also independent of each other. There can be no question about the independence of those which contain H and K. But the α-invariants of the first class contain derivatives of B as well as of A, and the β-invariants of the first class contain derivatives of A as well as of B. However, the derivatives of B in the α-invariants of order n are only of order $n-1$, so that no relation is possible between the α- and β-invariants.

The Wronskian, the U-process and the V-process suffice, therefore, to deduce all invariants from the four fundamental ones A, B, H and K. There are $2(n^2+2n+2)$ independent invariants of order equal to or less than n.

In order to find a system of covariants we investigate the result of the transformation

$$\bar{u} = \alpha(u), \qquad \bar{v} = \beta(v),$$

upon the four independent semi-covariants (40). We find

$$\bar{y} = y, \qquad \bar{z} = \frac{1}{\alpha_u}(z + \tfrac{1}{2}\eta y), \qquad \bar{\rho} = \frac{1}{\beta_v}(\rho + \tfrac{1}{2}\zeta y),$$

(68)

$$\bar{\sigma} = \frac{1}{\alpha_u \beta_v}(\sigma + \tfrac{1}{2}\eta\rho + \tfrac{1}{2}\zeta z + \tfrac{1}{4}\eta\zeta y).$$

It now becomes an easy matter to verify that the following four functions are covariants

(69) $\quad y, \qquad 6Az + A_u y, \qquad 6B\rho + B_v y, \qquad \sigma - a\rho - b'z + ab'y.$

They are clearly independent. *Every other covariant may be expressed in terms of the covariants* (69) *and of invariants.*

The most general transformation which leaves the canonical form invariant may be found as follows. We have shown that a system of differential equations of form (3) may be reduced to the canonical form

$$y_{uu} + 2by_v + fy = 0,$$

$$y_{vv} + 2a'y_u + gy = 0.$$

Make successively the transformations

$$y = \lambda\bar{y}, \qquad \bar{u} = \alpha(u), \qquad \bar{v} = \beta(v).$$

The resulting system will again be in the canonical form if, and only if,

$$\frac{\lambda_u}{\lambda} + \frac{1}{2}\frac{\alpha_{uu}}{\alpha_u} = 0, \qquad \frac{\lambda_v}{\lambda} + \frac{1}{2}\frac{\beta_{vv}}{\beta_v} = 0,$$

i. e., if

$$\lambda = \frac{g(v)}{\sqrt{\alpha_u}} = \frac{h(u)}{\sqrt{\beta_v}},$$

where $g(v)$ and $h(u)$ are functions of the single arguments indicated. We find, therefore,

$$\lambda = \frac{C}{\sqrt{\alpha_u \beta_v}},$$

where C is an arbitrary constant.

Therefore, *the most general transformation which leaves the canonical form invariant is*

(70) $\qquad \bar{y} = C\sqrt{\alpha_u \beta_v}\, y, \qquad \bar{u} = \alpha(u), \qquad \bar{v} = \beta(v),$

where α and β are arbitrary functions of u and v respectively, and where C is an arbitrary constant.

§ 8. *Principle of duality. Adjoined systems. The fundamental theorem.*

Given a system of partial differential equations in the intermediate form

$$\begin{aligned} y_{uu} + 2ay_u + 2by_v + cy &= 0, \\ y_{vv} + 2a'y_u + 2b'y_v + c'y &= 0, \end{aligned} \tag{71}$$

whose integrability conditions are fulfilled identically. If

$$y^{(k)} = f^{(k)}(u, v) \qquad (k = 1, 2, 3, 4),$$

are four independent solutions of (71), we interpret them as the homogeneous coördinates of a point P_y of a surface S. Let p_y be the plane which is tangent to S at P_y. Its equation will be

$$\begin{vmatrix} x_1 & x_2 & x_3 & x_4 \\ y' & y'' & y^{(3)} & y^{(4)} \\ y'_u & y''_u & y^{(3)}_u & y^{(4)}_u \\ y'_v & y''_v & y^{(3)}_v & y^{(4)}_v \end{vmatrix} = 0. \tag{72}$$

The coördinates of the plane p_y will be proportional to the minors of x_1, x_2, x_3, x_4 in this determinant. Denote these minors by $\lambda', \lambda'', \lambda^{(3)}, \lambda^{(4)}$. We shall have

$$\lambda' = \begin{vmatrix} y'' & y^{(3)} & y^{(4)} \\ y''_u & y^{(3)}_u & y^{(4)}_u \\ y''_v & y^{(3)}_v & y^{(4)}_v \end{vmatrix}, \qquad \lambda'' = -\begin{vmatrix} y' & y^{(3)} & y^{(4)} \\ y'_u & y^{(3)}_u & y^{(4)}_u \\ y'_v & y^{(3)}_v & y^{(4)}_v \end{vmatrix}, \text{ etc.}$$

Let us convert the columns into rows and write only a representative row. We may then write for each of the four λ's, disregarding signs, an expression of the form

$$\lambda = D(y, y_u, y_v), \tag{73}$$

which stands for a determinant of the third order of the matrix

$$|y^{(k)}, y^{(k)}_u, y^{(k)}_v| \qquad (k = 1, 2, 3, 4).$$

We find from (73), making use of (71),

$$\begin{aligned} \lambda_u &= D(y, y_u, y_{uv}) - 2a\lambda, \\ \lambda_v &= D(y, y_{uv}, y_v) - 2b'\lambda. \end{aligned} \tag{74}$$

Therefore, making use of (28) and (29),

$$\begin{aligned} \lambda_{uu} &= -2a\lambda_u + (4bb' - 2b_v - c - 2a_u)\lambda - 4aD(y, y_u, y_{uv}) - 2bD(y, y_v, y_{uv}), \\ \lambda_{vv} &= -2b'\lambda_v + (4aa' - 2a'_u - c' - 2b'_v)\lambda - 2a'D(y, y_{uv}, y_u) - 4b'D(y, y_{uv}, y_v). \end{aligned} \tag{75}$$

But from (74),

$$D(y,\ y_u,\ y_{uv}) = \lambda_u + 2a\lambda,$$
$$D(y,\ y_{uv},\ y_v) = \lambda_v + 2b'\lambda.$$

Substituting these values in (75) we find

$$\begin{aligned} &\lambda_{uu} + 6a\lambda_u - 2b\lambda_v + (2a_u + 2b_v + c + 8a^2 - 8bb')\lambda = 0, \\ &\lambda_{vv} - 2a'\lambda_u + 6b'\lambda_v + (2b'_v + 2a'_u + c' + 8b'^2 - 8aa')\lambda = 0. \end{aligned} \tag{76}$$

The coördinates of the plane p_y, therefore, satisfy the equations (76) *which are of the same form as* (71).

This system of equations (76) may be simplified. Consider the determinant,

$$\Delta = \begin{vmatrix} y'_{uv} & y'_u & y'_v & y' \\ y''_{uv} & y''_u & y''_v & y'' \\ y^{(3)}_{uv} & y^{(3)}_u & y^{(3)}_v & y^{(3)} \\ y^{(4)}_{uv} & y^{(4)}_u & y^{(4)}_v & y^{(4)} \end{vmatrix} = D(y_{uv}, y_u, y_v, y). \tag{77}$$

We find

$$\frac{\partial \Delta}{\partial u} = -4a\Delta, \qquad \frac{\partial \Delta}{\partial v} = -4b'\Delta. \tag{78}$$

But we have also

$$\frac{\partial a}{\partial v} = \frac{\partial b'}{\partial u},$$

according to the first equation of (30). We may put, therefore,

$$a = \frac{\partial p}{\partial u}, \qquad b' = \frac{\partial p}{\partial v},$$

so that (78) shows Δ to have the value

$$\Delta = Ce^{-4p}, \tag{79}$$

where C is a non-vanishing constant. In fact, if Δ were equal to zero there would be relations possible of the form

$$\alpha y^{(k)}_{uv} + \beta y^{(k)}_u + \gamma y^{(k)}_v + \delta y^{(k)} = 0 \qquad (k=1, 2, 3, 4),$$

with non-vanishing coefficients. But this is impossible in the case under consideration in which the general solution of system (71) depends upon four arbitrary constants.

The coördinates of the plane p_y were proportional to λ', λ'', $\lambda^{(3)}$, $\lambda^{(4)}$. Let us define them as being equal to

$$Y^{(k)} = \frac{\lambda^{(k)}}{\sqrt{\Delta}} \qquad (k=1, 2, 3, 4).$$

These four functions $Y^{(k)}$ will satisfy a system of differential equations obtained from (76) by the transformation

$$\lambda = \sqrt{\Delta}\, Y = e^{-2p} Y. \tag{80}$$

The resulting system of equations is

$$\begin{aligned} &Y_{uu} + 2aY_u - 2bY_v + (c + 2b_v - 4bb')Y = 0, \\ &Y_{vv} - 2a'Y_u + 2b'Y_v + (c' + 2a'_u - 4aa')Y = 0. \end{aligned} \tag{81}$$

This shall be called *the system adjoined to* (71). Denote its coefficients by $2\bar{a}$, $2\bar{b}$, etc. We have

$$\begin{aligned} &\bar{a} = a, \qquad \bar{b} = -b, \qquad \bar{c} = c + 2b_v - 4bb', \\ &\bar{a}' = -a', \qquad \bar{b}' = b', \qquad \bar{c}' = c' + 2a'_u - 4aa', \end{aligned}$$

whence

$$\begin{aligned} &a = \bar{a}, \qquad b = -\bar{b}, \qquad c = \bar{c} + 2\bar{b}_v - 4\bar{b}\bar{b}', \\ &a' = -\bar{a}', \qquad b' = \bar{b}', \qquad c' = \bar{c}' + 2\bar{a}'_u - 4\bar{a}\bar{a}', \end{aligned}$$

so that *the relation between* (71) *and* (81) *is reciprocal, i. e., each is the adjoint of the other.* The seminvariants of (81) are

$$\bar{a}',\ \bar{b},\ \bar{f},\ \bar{g},$$

which are respectively equal to the seminvariants

$$-a',\ -b,\ f,\ g$$

of (81).

System (81) *represents the same surface as* (71) *in plane- instead of in point-coördinates. But we may also interpret* Y', Y'', $Y^{(3)}$, $Y^{(4)}$ *as point-coördinates. In that case every integral surface of* (81) *would be dualistic to every integral surface of* (71).

It is clear, therefore, that *dualistic properties of any integral surface of* (71) *are analytically characterized by an invariant system of equations which remains unaltered if the signs of* a' *and* b *are changed.*

If we think of a surface as being at the same time described by a point and enveloped by its tangent planes we must consider the systems (71) and (81) simultaneously. A surface S may be *identically* self-dual; i. e., there may exist a dualistic transformation which converts it into itself in the particular way that every point P_y is converted into the plane p_y tangent to S at P_y and conversely. In that case (71) and (81) will be identical, so that

$$a' = b = 0.$$

The surface is then a quadric. In fact, we always have

$$y' Y' + y'' Y'' + y^{(3)} Y^{(3)} + y^{(4)} Y^{(4)} = 0,$$

and in this case $Y^{(k)}$ would have to be proportional to $y^{(k)}$ so that there will be a quadratic relation between $y' \cdots, y^{(4)}$. We may also see this in another way. If $a' = 0$, equations (71) show that the curves $u =$ const. are straight lines. If $b = 0$, the curves $v =$ const. are straight lines. But a quadric is the only surface which has two separate families of straight lines upon it.

We have, therefore, the following further result. *If either a' or b is equal to zero, the surface S is a ruled surface. If both a' and b vanish it is a quadric. Quadrics, moreover, are the only identically self-dual surfaces.*

We shall close this paper with the enunciation of a theorem which may be regarded as the *fundamental theorem* of the projective differential geometry of curved surfaces. Any non-developable surface being given

$$y^{(k)} = f^{(k)}(u, v) \qquad (k = 1, 2, 3, 4),$$

it determines a system of form (3) which may always be reduced to the intermediate or even the canonical form, from which the invariants a', b, h and k may be computed as functions of u and v. If on the other hand a', b, h and k are given as arbitrary functions of u and v, subject of course to the integrability conditions (34), the coefficients of the canonical form will be uniquely determined if a' and b are different from zero. In other words,

If the four invariants a', b, h, k are given as arbitrary functions of two variables u and v, subject to the integrability conditions (34), *and if moreover a' and b are not equal to zero, they determine a non-developable, non-ruled surface except for a projective transformation.*

It is not necessary to attempt to remove the restrictions of this theorem, as the author has already constructed a projective differential geometry of ruled and developable surfaces upon a different basis.

BERKELEY, *November* 21, 1906.

A METHOD FOR CONSTRUCTING THE FUNDAMENTAL REGION OF A DISCONTINUOUS GROUP OF LINEAR TRANSFORMATIONS*

BY

J. I. HUTCHINSON

Two methods at present exist for the construction of the fundamental region of a discontinuous group.† The one applies only to groups which are capable of extension by reflection and consists in constructing first the region for the extended group (which can be done by a direct process) and then combining two such regions that are adjacent. The other method uses the process of the continuous reduction of a quadratic form. It is, however, tedious in application, and this fact appears to place a narrow restriction on its range of practical effectiveness. In the following pages I propose a much more simple and effective method, applicable to groups which leave a given Hermitian form invariant.

§ 1. *Groups of transformations of a single variable.*

Let the complex variable be denoted by z and its conjugate by $\bar{z}$. Also let the equation of the fixed circle for the given group be

$$F \equiv \alpha z\bar{z} + \gamma z + \bar{\gamma}\bar{z} + \beta = 0, \tag{1}$$

in which the discriminant $\delta = \gamma\bar{\gamma} - \alpha\beta$ is positive (α, β being real) and α is negative. The circle is then real and its interior will be defined by the inequality $F > 0$.

Let $F = 0$ be transformed into itself by the substitution

$$z' = \frac{az + b}{cz + d}, \qquad ad - bc = 1. \tag{2}$$

The conditions for this are

$$\begin{aligned} &\alpha(a\bar{a} - 1) + \gamma a\bar{c} + \bar{\gamma}\bar{a}c + \beta c\bar{c} = 0, \\ &\alpha a\bar{b} + \gamma a\bar{d} + \bar{\gamma}c\bar{b} \times \beta c\bar{d} = \gamma, \\ &\alpha b\bar{b} + \gamma b\bar{d} + \bar{\gamma}\bar{b}d + \beta(d\bar{d} - 1) = 0. \end{aligned} \tag{3}$$

Since $F = 0$ is also transformed into itself by the inverse of (2), the conditions

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† See FRICKE-KLEIN, *Automorphe Functionen*, I, p. 539.

(3) may be replaced by the equivalent ones,

$$\begin{aligned} \alpha(d\bar{d}-1)-\gamma d\bar{c}-\bar{\gamma}\bar{d}c+\beta c\bar{c}&=0,\\ -\alpha d\bar{b}+\gamma d\bar{a}+\bar{\gamma}c\bar{b}-\beta c\bar{a}&=\gamma,\\ \alpha b\bar{b}-\gamma b\bar{a}-\bar{\gamma}a\bar{b}+\beta(a\bar{a}-1)&=0. \end{aligned} \tag{3'}$$

The transformation (2) changes the function F into F' such that

$$F'=F/\Delta\bar{\Delta}, \tag{4}$$

$$\Delta=cz+d. \tag{5}$$

Suppose that for a given point z inside the circle (1) we have $\Delta\bar{\Delta}<1$; thence follows $F'>F$, and the effect of the substitution has been to increase the value of the positive function F. If there exists a second substitution changing the point z' into z'' so that $F''>F'$, we apply it to the preceding one and so proceed until F has reached its maximum value for the series of congruent points $z, z', z'', \cdots$. Such a maximum value will be attained after a finite number of steps provided that the substitutions employed all belong to a group G which is "properly discontinuous" in the z-plane. I will name the point so determined *the maximum point congruent to* z.

There may be several points congruent to z for which F has a maximum value. The number of these is equal to the order* ν of the subgroup g of G defined by $c=0$, since under this condition it follows from (3') that $\Delta\bar{\Delta}=1$. The totality of maximum congruent points obtained by varying z throughout the interior of F will form a region R bounded by certain circles, orthogonal to $F=0$, whose equations are

$$\Delta\bar{\Delta}=1. \tag{6}$$

The region R is divided by means of g into ν congruent regions $R_1, R_2, \cdots, R_\nu$. Any one of the regions R_i may then be taken as the fundamental region for the given group G.

It only remains to show how to select from among the circles (6) those which bound the region R. For this purpose we observe that when F increases, the point z approaches the center C of the circle $F=0$. Accordingly the boundaries of R are to be selected from those Δ-circles (6) which pass nearest to C. The point of $\Delta\bar{\Delta}=1$ nearest to C is

$$z=\frac{d\lambda+\sqrt{\lambda\bar{\lambda}}}{-c\lambda}, \tag{7}$$

in which

$$\lambda=\gamma\bar{c}-\alpha\bar{d}. \tag{8}$$

* This order may be infinite as in the case of the modular group when it has the real axis for its fixed circle.

The value of F at this point is

$$F = 2(\alpha + \sqrt{\lambda\bar{\lambda}})/c\bar{c}. \tag{9}$$

This is evidently the greatest value attained by F on the given Δ-circle, and it may be found by the usual process for determining the maximum value of a function.

From (8) we deduce

$$\gamma\bar{\gamma}c\bar{c} = \alpha\gamma\bar{c}d + \alpha\bar{\gamma}c\bar{d} - \alpha^2 d\bar{d} + \lambda\bar{\lambda};$$

also, from the first equation (3′),

$$\alpha\beta c\bar{c} = \alpha\gamma d\bar{c} + \alpha\bar{\gamma}c\bar{d} - \alpha^2 d\bar{d} + \alpha^2,$$

whence, by subtraction,

$$\delta c\bar{c} = \lambda\bar{\lambda} - \alpha^2.$$

Using this result in (9) we have for the maximum value of F on (6),

$$F = \frac{2\delta}{\sqrt{\lambda\bar{\lambda}} - \alpha}.$$

From this it follows that the greatest values attained by F at the points defined by (7) are those for which the absolute values of λ are the least.

It is evident, then, that the boundary circles for the region R occur among those determined by the smallest values of $|\lambda|$. There does not appear to be a sufficiently simple and general criterion by which to select out the required circles from among those found in this way, but a graphical construction will quickly indicate which ones are to be used.

The coefficients a, b, d are expressible in terms of λ, c in the form

$$a = \frac{\lambda + \bar{\gamma}c}{-\alpha}, \qquad d = \frac{\bar{\gamma}c - \bar{\lambda}}{\alpha}, \qquad b = \frac{\bar{\gamma}(\bar{\lambda} - \bar{\gamma}c) + \delta\bar{c} - \lambda\bar{\gamma}}{\alpha^2}, \tag{10}$$

the numbers λ, c being connected by the relation

$$\lambda\bar{\lambda} = \alpha^2 + \delta c\bar{c} \tag{11}$$

which is the condition that the determinant shall equal 1. Equation (11) is also the condition that the circle $\Delta\bar{\Delta} = 1$ shall be orthogonal to $F = 0$.

To give a clearer idea of how the preceding method works out in practise, we consider the following examples in which a, b, c, d are restricted to complex integers.

Example I. $\alpha = 1, \gamma = 0, \beta = -5$.

For $c = 0$ a substitution of period 2 is obtained having the origin as fixed point. Hence the region R consists of two congruent regions R_1, R_2. For $|c| > 0$ the minimum value of $\lambda\bar{\lambda}$ is 26, and hence $\lambda = \pm 5 \pm i$ or $\pm 1 \pm 5i$;

also $c\bar{c} = 5$ and therefore $c = \pm 2 \pm i$ or $\pm 1 \pm 2i$. The equation of the Δ-circle is $(cz - \bar{\lambda})(\bar{c}\bar{z} - \lambda) = 1$, which may be written

$$\left(z - \frac{\bar{\lambda}}{c}\right)\left(\bar{z} - \frac{\lambda}{\bar{c}}\right) = \frac{1}{c\bar{c}},$$

and hence its center is the point $z = \bar{\lambda}/c$ and its radius is $1/\nu\, c\bar{c}$. With the values just obtained for λ, c we construct the corresponding circles $1, 1, \cdots$, Fig. 1, using only those which occur in the upper half plane in which we take

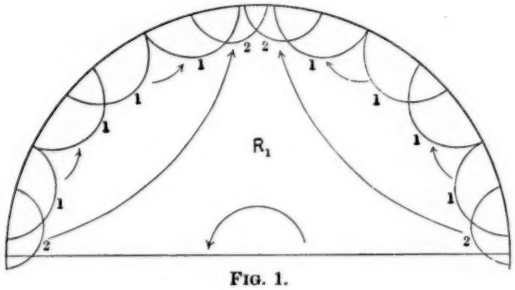

FIG. 1.

R_1. These do not form a closed region and we accordingly take the next higher value of $\lambda\bar{\lambda}$ which is 41, whence $c\bar{c} = 8$. The circles $2, 2, \cdots$ are determined by these values, and now the region above the real axis being closed we have in R_1 thus determined the fundamental region for the given group.*

Example II. $\alpha = -3, \beta = \gamma = 1$.

It is useful to observe from the relation $\lambda + \bar{\lambda} = -\alpha(a + d)$ that $\lambda + \bar{\lambda}$ is divisible by α. For $c = 0$ the only substitution of the group is identity. The values of λ and c first to be considered are those for which $\lambda\bar{\lambda} = 13$, $c\bar{c} = 1$. Among these the only ones which give different substitutions with complex integral values for a, b, d are $\lambda = 3 + 2\epsilon i$, $c = \epsilon i$, $\epsilon = \pm 1$. These determine the two circles 1, 2, Fig. 2. The next higher values of λ and c are derived from $\lambda\bar{\lambda} = 25$, $c\bar{c} = 4$, whence we obtain the circles 3, 4, 5. We exclude 4, 5 since they lie inside 1, 2. As the region about the center C is not yet closed we take the next case $\lambda\bar{\lambda} = 45$, $c\bar{c} = 9$, which determines the circles 6, 7, 8, 9, the last two of which are to be excluded. As the region R is now closed the operation is completed.

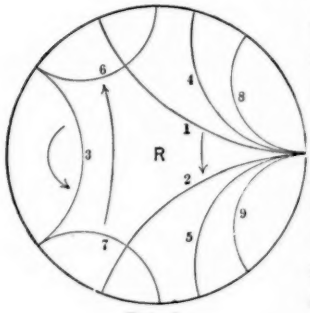

FIG. 2.

*Cf. FRICKE-KLEIN, *Automorphe Functionen*, I, p. 479.

§ 2. *Transformations on two variables.*

It is evident that the general method explained above is at once applicable to groups of substitutions on any number of variables, the region R being then defined by certain inequalities of the form $\Delta\bar{\Delta} > 1$. For the sake of simplicity I will confine myself in the present paper to the case of a group G of linear transformations on two complex variables x, y, the general substitution being written in the form

$$x' = (ax + by + c)/\Delta, \qquad y' = (dx + ey + f)/\Delta \tag{12}$$

in which $\Delta = px + qy + r$. I will further suppose that the group G leaves the hypersphere

$$F \equiv 1 - x\bar{x} - y\bar{y} = 0 \tag{13}$$

invariant. The conditions on the coefficients are

$$\begin{aligned} a\bar{a} + d\bar{d} - p\bar{p} &= 1, & a\bar{b} + d\bar{e} - p\bar{q} &= 0, \\ b\bar{b} + e\bar{e} - q\bar{q} &= 1, & a\bar{c} + d\bar{f} - p\bar{r} &= 0, \\ c\bar{c} + f\bar{f} - r\bar{r} &= -1, & b\bar{c} + e\bar{f} - q\bar{r} &= 0. \end{aligned} \tag{14}$$

Multiply the two equations (12) by $\bar{a}$, $\bar{d}$ respectively and add; then subtract $\bar{p}$ from both members of the resulting equation and use (14). This gives

$$\bar{a}x' + \bar{d}y' - \bar{p} = x/\Delta;$$

similarly,

$$\bar{b}x' + \bar{e}y' - \bar{q} = y/\Delta \qquad \text{and} \qquad \bar{c}x' + \bar{f}y' - \bar{r} = -1/\Delta.$$

Hence the inverse of (12) may be written

$$x = (\bar{a}x' + \bar{d}y' - \bar{p})/\Delta', \qquad y = (\bar{b}x' + \bar{e}y' - \bar{q})/\Delta', \tag{15}$$

$$\Delta' = -\bar{c}x' - \bar{f}y' + \bar{r}.$$

As the inverse substitution also leaves the hypersphere invariant we have the conditions

$$\begin{aligned} a\bar{a} + b\bar{b} - c\bar{c} &= 1, & a\bar{d} + b\bar{e} - c\bar{f} &= 0, \\ d\bar{d} + e\bar{e} - f\bar{f} &= 1, & a\bar{p} + b\bar{q} - c\bar{r} &= 0, \\ p\bar{p} + q\bar{q} - r\bar{r} &= -1, & d\bar{p} + e\bar{q} - f\bar{r} &= 0. \end{aligned} \tag{14'}$$

From the fourth and fifth of these equations we deduce

$$a = k(\bar{f}\bar{q} - \bar{e}\bar{r}), \qquad b = k(\bar{d}\bar{r} - \bar{f}\bar{p}), \qquad c = k(\bar{d}\bar{q} - \bar{e}\bar{p}).$$

In order to evaluate the unknown k, we substitute these expressions for a, b, c

in the determinant

$$\begin{vmatrix} a & b & c \\ d & e & f \\ p & q & r \end{vmatrix} = N$$

and obtain

$$k(-A\bar{A} - B\bar{B} + C\bar{C}) = N,$$

in which A, B, C are the minors corresponding to a, b, c. But since the inverse of the substitution (12) is

$$x = \frac{Ax' + Dy' + P}{Cx' + Fy' + R}, \qquad y = \frac{Bx' + Ey' + Q}{Cx' + Fy' + R},$$

it follows that $A\bar{A} + B\bar{B} - C\bar{C} = 1$ and therefore $k = -N$. Hence,

$$\begin{aligned} a &= N(\bar{e}\bar{r} - \bar{f}\bar{q}) = N\bar{A}, \\ b &= N(\bar{f}\bar{p} - \bar{d}\bar{r}) = N\bar{B}, \\ c &= N(\bar{e}\bar{p} - \bar{d}\bar{q}) = -N\bar{C}, \end{aligned} \tag{16}$$

in which $d, e, \cdots$ satisfy the three relations

$$\begin{aligned} p\bar{p} + q\bar{q} - r\bar{r} &= -1, \\ d\bar{d} + e\bar{e} - f\bar{f} &= 1, \\ d\bar{p} + e\bar{q} - f\bar{r} &= 0. \end{aligned} \tag{17}$$

We next proceed to determine the maximum point of the *spread*

$$\Delta\bar{\Delta} = 1, \qquad \Delta = px + qy + r, \tag{18}$$

that is, the point at which the $F = 1 - x\bar{x} - y\bar{y}$ takes its maximum value when x, y are restricted to the locus (18). Regarding $\bar{y}$ as a dependent variable on account of (18) we differentiate F with respect to $x, \bar{x}, y$ and obtain as conditions for a maximum

$$\bar{x} + y\frac{\partial\bar{y}}{\partial x} = 0, \qquad x + y\frac{\partial\bar{y}}{\partial\bar{x}} = 0, \qquad \bar{y} + y\frac{\partial\bar{y}}{\partial y} = 0.$$

By differentiating $\Delta\bar{\Delta} = 1$ and substituting the expressions thus obtained for the derivatives, we deduce

$$x = \frac{\bar{p}}{\bar{q}}y, \qquad \bar{x} = \frac{p\bar{\Delta}}{\bar{q}\Delta}y, \qquad \bar{y} = \frac{q\bar{\Delta}}{\bar{q}\Delta}y. \tag{19}$$

By substitution of these results in the second equation (18) we obtain

$$\Delta = (r\bar{r} - 1)\bar{y}/\bar{q},$$

and by substituting in (10),

$$\bar{y} = \frac{q\bar{r}}{\bar{q}r}y, \qquad \bar{x} = \frac{p\bar{r}}{\bar{q}r}y, \qquad x = \frac{\bar{p}}{\bar{q}}y.$$

Substitute now in $\Delta\bar{\Delta} = 1$ and solve for y. This gives

$$y = \frac{\bar{q}(-rr \pm \sqrt{r\bar{r}})}{\bar{r}(r\bar{r}-1)}.$$

Therefore

$$F = \frac{-2 \pm 2\sqrt{r\bar{r}})}{r\bar{r}-1}.$$

From the first equation (17) it is evident that $r\bar{r}$ cannot be less than 1. In order that the point determined may be inside the hypersphere it is necessary to take the lower sign. Hence, *the point*

$$x = \frac{\bar{p}(\sqrt{r\bar{r}} - r\bar{r})}{\bar{r}(r\bar{r}-1)}, \qquad y = \frac{\bar{q}(\sqrt{r\bar{r}} - r\bar{r})}{\bar{r}(r\bar{r}-1)}$$

is the maximum point for the spread $\Delta\bar{\Delta} = 1$ *and gives to* F *the value*

$$F - \frac{2}{\sqrt{r\bar{r}}+1}. \tag{20}$$

A glance at (20) shows that the largest maximum values are attained by F when $r\bar{r}$ has its smallest values. The corresponding Δ-spreads are those of (18) which pass nearest to the center of $F = 0$, and the boundaries for the region R are to be selected from among these.

§ 3. *Klein's group of* $(n+1)!$ *collineations.*

The working out of the fundamental region for an infinite discontinuous group of the class just treated appears to involve a considerable amount of necessary detail and for that reason I reserve for a future paper the discussion of particular groups. In the present article I will only notice, by way of illustration, the fundamental region R for KLEIN's group of $(n+1)!$ collineations defined by the homogeneous equations of transformation on the variables y_i,

$$y_i' = y_{a_i} \qquad (i = 1, 2, \cdots, n+1), \tag{21}$$

in which $a_1, a_2, \cdots, a_{n+1}$ is any permutation of the numbers $1, 2, \cdots, n+1$. We first make the group non-homogeneous by introducing the variables $x_k = y_k/y_{n+1}$ $(k = 1, 2, \cdots, n)$. Equations (21) then become

$$x_k' = x_{a_k}/x_i \qquad (k = 1, 2, \cdots, n), \tag{22}$$

in which x_{a_k} of the numerator is to be replaced by 1 when $a_k = i$. We have, then, $\Delta = x_i$ and the inequality conditions $\Delta\bar{\Delta} > 1$ reduce to

$$x_i\bar{x}_i - 1 > 0 \qquad (i=1, 2, \cdots, n). \tag{23}$$

Apply now to (23) the different transformations of the group. We obtain additional inequalities of the form

$$x_i\bar{x}_i - x_j\bar{x}_j > 0 \qquad (i, j=1, 2, \cdots, n). \tag{24}$$

A complete set of consistent inequalities of this type will be obtained by subjecting i and j to the condition $i > j$.

Returning now to the homogeneous variables, the conditions (23) and (24) become

$$y_i\bar{y}_i - y_j\bar{y}_j > 0, \qquad i > j \qquad (i, j=1, 2, \cdots, n+1). \tag{25}$$

These inequalities (some of which are superfluous) define a fundamental region R for the given group. For, since any substitution of the group produces a permutation of the y_i, it follows that at least one of the inequalities (25) is reversed. The only case in which this is not immediately apparent is that of the substitution which produces the cyclic permutation $(y_1, y_2, \cdots, y_{n+1})$. The inequalities (25) are permuted except that in place of $y_2\bar{y}_2 - y_1\bar{y}_1 > 0$ we now have $y_1\bar{y}_1 - y_{n+1}\bar{y}_{n+1} > 0$. But if we add this last to the $n-1$ inequalities $y_{i+1}\bar{y}_{i+1} - y_i\bar{y}_i > 0 (i = 2, 3, \cdots, n)$, we obtain $y_1\bar{y}_1 - y_2\bar{y}_2 > 0$, which is a reversal of the first inequality (25). Hence every point within R is transformed into a point outside R by each substitution of the group. On the other hand, every point outside R can be transformed into a point within. For, the coördinates of an arbitrarily fixed point can evidently satisfy a set of conditions of the form

$$y_{a_i}\bar{y}_{a_i} - y_{a_j}\bar{y}_{a_j} > 0, \qquad i > j \qquad (i, j=1, 2, \cdots, n+1). \tag{26}$$

By choosing the a_i as all possible permutations of $1, 2, \cdots, n+1$ we obtain $(n+1)!$ different sets of conditions (26), including (25). It is also clear that no other set of conditions except those enumerated would be a consistent one. For if in the $n(n+1)$ inequalities (26) no variable occurred in the first term in more than n of the formulas, it would be necessary for two of the inequalities to have the form $y_k\bar{y}_k - y_l\bar{y}_l > 0$, $y_l\bar{y}_l - y_k\bar{y}_k > 0$, which are contradictory. Suppose, then, that y_{a_1} occurs in the first term of $n+1$ of the inequalities (26). A similar argument will show that some other variable, say y_{a_2}, must occur in the first term of n of the remaining inequalities, and so on. We accordingly have the arrangement of (26). But since there are $(n+1)!$ substitutions in the group and no substitution, except 1, leaves a particular set unaltered, it follows that any given set (26) can be transformed into (25) by a certain substitution of the group, and therefore any point outside R has a congruent point inside.

After excluding superfluous conditions it is evident that the $n(n+1)$ inequalities (25) reduce to *

$$y_{i+1}\bar{y}_{i+1} - y_i\bar{y}_i > 0 \qquad (i=1, 2, \cdots, n). \tag{27}$$

The boundary

$$y_i\bar{y}_i - y_{i+1}\bar{y}_{i+1} = 0 \tag{28}$$

is divided by the flat space $y_i = y_{i+1}$ into two parts which are interchanged by the substitution

$$y'_i = y_{i+1}, \qquad y'_{i+1} = y_i, \qquad y'_k = y_k \quad (k=1, 2, \cdots, i-1, i+2, \cdots, n+1),$$

while the $n-1$ remaining inequalities (27) are unaltered. Hence each of the boundaries (28) is paired with itself.

* The fundamental region for this group, when the variables y_i are restricted to real values, is given by E. H. MOORE, *Concerning Klein's group of* $(n+1)!$ *n-ary collineations*, American Journal of Mathematics, vol. 22 (1900), p. 336. When the variables are real the conditions (27) become $y_{i+1}^2 - y_i^2 > 0$. For positive values of the y's these reduce to exactly the conditions given by Professor Moore, p. 338; for negative values of the variables the two sets of conditions do not coincide, but they are equivalent and one may be reduced to the other by an "allowable change" of the fundamental region.

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OBLIQUE REFLECTIONS AND UNIMODULAR STRAINS*

BY

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Introduction

The object of the present communication is to study the group of unimodular strains about a fixed point in its relation to products of involutory strains.† The interest in this group is not merely geometric; rather is it mechanical. The group of strains is fundamental in the theory of elasticity and as such has long been studied by students of both theoretical and applied mathematics. Indeed the different types of strain, such as simple and complex shears, tonics and cyclotonics, have been carefully classified‡ so that the present investigation has something well known to which to attach itself. The restriction to unimodular or equivoluminal strains is necessitated by the fact that an involutory strain obviously cannot alter (except for sign) the volume of any portion of space, and hence the most general strain that can be obtained as a product of successive involutory strains must leave volume unchanged in magnitude. It may be worth while to note that the ether is a body which possesses this unimodular elastic property, and that the instantaneous strain at a point in an incompressible fluid is similarly characterized.

As the use of multiple algebras in geometric investigations has not been very wide spread, it may be proper at this juncture to enter somewhat upon the question of what algebra should be chosen for the present purpose. Four, at least, suggest themselves immediately. They are: GRASSMANN's point analysis, PEANO's vector analysis, quaternions, and GIBBS's vector analysis.§ Any of these,

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† Geometric investigations on the relation of involutory transformations to various groups in which they occur have been carried on during the last fifteen years by numerous authors. For a general résumé of this work see an article on *Involutory transformations in the projective group and in its subgroups* by the present writer in the Annals of Mathematics, ser. 2, vol. 7 (1907), pp. 77–86.

‡ See, for example, THOMSON AND TAIT, *Natural Philosophy* (new edition), vol. 1, pt. 1, pp. 139–185. Also GIBBS, *Elements of Vector Analysis* (1881–84); *Scientific Papers*, vol. 2 (1906), pp. 17–90.

§ For GRASSMAN's analysis one may consult his first *Ausdehnungslehre* (1844). MEHMKE, *Vergleich der Vektoranalysis americanischer Richtung und derjenigen deutsch-italienischer Richtung*, Jahresberichte der deutschen Math.-Ver., vol. 13 (1904), pp. 217–228, gives some

and others as well, might serve. The first two are more truly projective or strain algebras than the last two. They do not depend so essentially on relations of perpendicularity, which of course are not preserved by strains. Hence they would appear more germane to the subject in hand.* The physicists, however, make great use of relations of perpendicularity in connection with the study of strains — the strain invariants are estimated on perpendicular axes, the strain quadric is referred to such axes, and so on. In short it does appear desirable to take account of perpendicularity.† This is done alike by quaternions and the vector analysis of GIBBS. The former seems nevertheless to be not quite so well suited to the questions in hand as is the latter; for the linear vector function of the quaternion analysis is not treated primarily as an independent multiple quantity with an independent set of algebraic relations but rather as an operator meaningless apart from the vector to which it is applied. Moreover strains have been carefully treated in GIBBS's work‡ and the treatment is perhaps more easily available for reference than in any other which uses multiple algebra. Hence, everything considered, GIBBS's algebra will be chosen in this instance, although any one of the others would doubtless exhibit at some stage of the work its peculiar advantages over the rest.

It may be taken as known that there are three and only three types of real involutory strains:§ 1°, a strain which reverses in direction every vector issuing from the origin; 2°, a strain which reverses in direction vectors parallel to a given line and leaves unchanged vectors parallel to a given plane which does not contain the line; 3°, a strain which reverses in direction vectors parallel to a given plane and leaves unchanged vectors parallel to a given line which does not lie in the plane. The first of these three types of involutory strain will be called *central reflection;* the second, *planar reflection* parallel to a line; the third, *linear reflection* parallel to a plane. The last two types are, so to speak, *dual* in nature — one through a plane parallel to a line, the other through a line parallel to a plane.

These transformations may be expressed as dyadics.‖ Let the dyadics be

account of the methods of PEANO. See also PEANO, *Formulario Matematico* (editio v) (1906), pp. 165–201. The real difficulty with these algebras is not so much their inappropriateness as the undeveloped state of their linear vector function.

* For this side of the question consult MEHMKE, loc. cit.

† For this side of the question consult PRANDTL, *Ueber die physikalische Richtung in der Vektoranalysis,* Jahresberichte etc, vol. 13 (1904), pp. 436–449. There are some general remarks on both sides of the question by myself in the *Verhandlungen des* III. *internationalen Mathematiker-Kongresses* (1904), pp. 202–215.

‡ In the pamphlet referred to above and more especially in his *Vector Analysis* edited by myself (1901), chap. 6 (Charles Scribner's Sons, New York). This work will be referred to as *Vector Analysis* in the remaining footnotes or portions of the text.

§ For a full discussion of the types of involutory strains in the projective group and its subgroups see my *Involutory transformations,* etc., previously cited.

‖ *Vector Analysis,* p. 332.

used as prefactors so that

$$\rho' = \Phi \cdot \rho$$

represents the transformation of the position vector ρ into the position vector ρ'. Consider the dyadic Φ to be expressed in terms of any three noncomplanar vectors α, β, γ and their reciprocals α', β', γ' as antecedents and consequents respectively. The identical transformation is represented by the idemfactor which in this case takes the form *

$$I = \alpha\alpha' + \beta\beta' + \gamma\gamma'.$$

Central reflection which merely effects a reversal of all directions is represented by the negative of the idemfactor, that is by

$$-I = -(\alpha\alpha' + \beta\beta' + \gamma\gamma'),$$

and this is the only way in which it may be represented so long as the antecedents are to form a set reciprocal to the consequents.

For oblique planar reflection in the plane of α, β and along the direction of γ, the dyadic

$$\Phi = \alpha\alpha' + \beta\beta' - \gamma\gamma' = I - 2\gamma\gamma'$$

evidently suffices. The analytical form of the dyadic will not be nearly so simple unless the vectors α, β, γ are chosen in the plane and along the direction of the reflection. There, is however, a form which at least in appearance is a little more general than the above and which we will adopt in the future. Assume for trial the form

$$\Phi = I - 2\epsilon\zeta.$$

Any vector ρ perpendicular to ζ becomes

$$\rho' = \Phi \cdot \rho = I \cdot \rho - 2\epsilon\zeta \cdot \rho = \rho.$$

Hence the plane perpendicular to ζ and all components parallel to this plane are left unchanged. Any vector parallel to ϵ becomes

$$\rho' = \Phi \cdot x\epsilon = (1 - 2\zeta \cdot \epsilon)x\epsilon,$$

and this amounts to reversal of direction when and only when $\zeta \cdot \epsilon = 1$.

Hence the general form of a dyadic which shall represent reflection in a plane perpendicular to ζ and along the direction ϵ is

$$\Phi = I - 2\epsilon\zeta, \qquad \zeta \cdot \epsilon = 1.$$

If for any reason some other form of the dyadic should be desired, it is only necessary to express I, ϵ, ζ in terms of any convenient antecedents and consequents.

In like manner a simple inspection shows that the dyadic

* *Vector Analysis*, p. 288, et seq.

$$\Phi = -\alpha\alpha' - \beta\beta' + \gamma\gamma' = 2\gamma\gamma' - I$$

represents an oblique linear reflection through the line whose direction is γ and parallel to the plane of α, β. This admits of the more general form

$$\Phi = 2\epsilon\zeta - I, \qquad \epsilon \cdot \zeta = 1.$$

It is noteworthy that both the linear and the planar reflections have a fundamental *line* and *plane*, although the part played by the line and plane is related in a sort of dual fashion in the two cases. To sum up the essential points in the analysis we state

THEOREM 1. *The three forms of reflection in the group of strains with their corresponding dyadics are*

$$\begin{aligned} &\textit{central}, && \Phi = -I, \\ &\textit{planar}, && \Phi = I - 2\epsilon\zeta, && \epsilon \cdot \zeta = 1, \\ &\textit{linear}, && \Phi = 2\epsilon\zeta - I, && \epsilon \cdot \zeta = 1, \end{aligned}$$

where ϵ represents the line that enters into the reflection whether the reflection takes place along it or through it, and where ζ is perpendicular to, that is, represents the plane of the reflection whether the reflection takes place through this plane or parallel to it.*

The analytical forms given above have been obtained from the known geometric properties of the reflections which have been proved to be the sole involutory strains. If it were not the intention to make use of the general theory of involutory collineations before entering upon the detail of this paper, the special case of involutory strains which concerns us might have been treated directly as an algebraic problem, and in the following way. Let Φ represent any involutory strain. Then

$$\rho' = \Phi \cdot \rho, \qquad \rho = \Phi \cdot \rho' = \Phi \cdot \Phi \cdot \rho = \Phi^2 \cdot \rho.$$

Hence, as the relation

$$\rho = I \cdot \rho = \Phi^2 \cdot \rho$$

holds for all vectors ρ, it follows that

$$I = \Phi^2.$$

In other words the geometric problem of determining all involutory strains is coextensive with the algebraic problem of finding all dyadics Φ which satisfy the equation $\Phi^2 = I$, that is, which are square roots of the idemfactor.

To solve the equation $\Phi^2 = I$, assume the solution in the form

$$\Phi = \Psi + I.$$

Then

$$I = \Phi^2 = (\Psi + I) \cdot (\Psi + I) = \Psi \cdot \Psi + 2\Psi \cdot I + I.$$

* *Vector Analysis*, p. 46.

Hence

$$\Psi\cdot\Psi + 2\Psi\cdot I = \Psi\cdot(\Psi + 2I) = 0.$$

Now if the product of two dyadics is zero one of them must be zero or one of them must be linear while the other is at most merely planar.* If $\Psi = 0$, then $\Phi = I$, which is involutory although trivially so. If $\Psi + 2I = 0$, then $\Phi = -I$, which is the central reflection. Next suppose that Ψ is linear and write it as $-2\epsilon\zeta$. Then

$$\Phi = I - 2\epsilon\zeta \qquad \text{and} \qquad \Psi + 2I = 2(I - \epsilon\zeta).$$

The latter must be planar. The condition of planarity is that the third of the dyadic vanish. Hence

$$(I - \epsilon\zeta)_3 = I_3 - \epsilon\cdot I\cdot\zeta = 1 - \epsilon\cdot\zeta = 0,\dagger$$

or $\epsilon\cdot\zeta = 1$ as before. In like manner if $\Psi + 2I$ were linear, assume

$$\Psi + 2I = +2\epsilon\zeta, \qquad \Phi = 2\epsilon\zeta - I.$$

Then $\Psi = 2(\epsilon\zeta - I)$ must be planar and the application of the condition shows that $\epsilon\cdot\zeta = 1$ again in this case. Hence the problem has been solved by purely algebraic means. The only square roots of the idemfactor are of the form

$$\pm I, \qquad \pm(I - 2\epsilon\zeta) \text{ with } \epsilon\cdot\zeta = 1.$$

The dyadic $\Phi_2 = \frac{1}{2}\Phi^{\times}_{\times}\Phi$ gives the transformation of surfaces.‡ If Φ_2 be calculated it is seen to be

$$\Phi_2 = I \qquad \text{if} \qquad \Phi = \pm I,$$

or

$$\Phi_2 = 2\zeta\epsilon - I \qquad \text{if} \qquad \Phi = \pm(I - 2\epsilon\zeta).$$

The change of volume is given by Φ_3. The calculation§ of this shows that volumes are reversed in sign by central or planar reflection, but are left unchanged by linear reflection. The results may be stated as

THEOREM 2. *In central reflection surfaces (regarded as vectors) are unchanged while volumes change sign. In planar and linear reflection surfaces (regarded as vectors) suffer linear reflection through a line perpendicular to the plane of the original reflection Φ and along a plane perpendicular to the line of the reflection Φ. Volumes change or keep their sign according as Φ is planar or linear.*

* *Vector Analysis*, pp. 282–288. This is but a particular case of a theorem in the general theory of matrices and due to SYLVESTER, namely: The nullity of a product of two matrices is not less than the greater of the nullities of the factors nor greater than the sum of the nullities of both factors. See WHITEHEAD, *Universal Algebra*, Vol. 1 (1899), p. 253.

† *Vector Analysis*, p. 331, Exs. 19 and 20. These formulas are important.

‡ *Vector Analysis*, p. 333. The best method of calculating ϕ_2 in this case is by the relation (68) of p. 313; for as ϕ is involutory, $\phi_2 = \pm\phi_c^{-1}$.

§ *Vector Analysis*, p. 333. The value of ϕ_3 may be calculated as above.

1. *The special case in the composition of two reflections.*

One of the first questions to arise is: When are two reflections commutative? It is obvious that whatever be the dyadic Φ,

$$-I\cdot\Phi = \Phi\cdot(-I) = -\Phi$$

and hence that central reflection is commutative not merely with other reflections, but with any strain whatsoever. Consider therefore two reflections neither of which is central. They may be written as

$$\pm(I-2\epsilon\zeta) \qquad \text{and} \qquad \pm(I-2\eta\xi),$$

where the sign is plus if the reflection is planar, and minus if it is linear. The condition that the reflections be commutative is

$$\pm(I-2\epsilon\zeta)\cdot(I-2\eta\xi) = \pm(I-2\eta\xi)\cdot(I-2\epsilon\zeta),$$

which, on expansion, reduces to the equation

$$\zeta\cdot\eta\epsilon\xi = \xi\cdot\epsilon\eta\xi.$$

This equation is satisfied, first if ϵ and η are parallel (as they both may be considered as passing through the origin, they may be taken as actually collinear) and ζ and ξ are parallel at the same time; second, if $\zeta\cdot\eta = 0$ and $\zeta\cdot\epsilon = 0$. The latter case is obvious, and the former becomes so when it is remembered that a single dyad possesses but five degrees of freedom, of which four are here satisfied by the relations of collinearity and the fifth by the equality in magnitude which results from the homogeneity of the equation. The geometric relation of the reflections in the first case needs no discussion; in the second case the equation $\zeta\cdot\eta = 0$ shows that η is perpendicular to ζ. Now ζ represents the plane of the reflection. Hence the line η lies in the plane perpendicular to ζ, that is, in the plane of the reflection. To sum up:

THEOREM 3. *That two reflections be commutative it is necessary and sufficient that:* 1°, *one of them be central;* 2°, *their lines and planes be respectively coincident; or* 3°, *the lines of each lie in the plane of the other.*

To find the result of the product of two commutative reflections note first that the product of a central reflection into a planar or linear reflection gives the other one of the two. For the product of two reflections, neither of which is central, we have

$$\pm(I-2\epsilon\zeta)\cdot(I-2\eta\xi) = \pm(I-2\epsilon\zeta-2\eta\xi+4\zeta\cdot\eta\epsilon\xi).$$

In case ϵ and ζ are respectively parallel to η and ξ they may be taken equal to them, for the relations $\epsilon\cdot\zeta = \eta\cdot\xi = 1$ show that the angle between the pairs of vectors ϵ, ζ and η, ζ are the same and hence that the vectors are either equal or opposite, which amounts to the same thing in this problem. Hence the product

is $\pm I$. In case $\zeta\cdot\eta = \xi\cdot\epsilon = 0$ the product is

$$\pm (I - 2\epsilon\zeta - 2\eta\xi)$$

with the relations

$$\epsilon\cdot\zeta = 1, \qquad \eta\cdot\xi = 1, \qquad \epsilon\cdot\xi = 0, \qquad \zeta\cdot\eta = 0.$$

Hence

$$\zeta \times \xi\cdot\epsilon \times \eta = \zeta\cdot\epsilon\xi\cdot\eta - \xi\cdot\epsilon\xi\cdot\eta = 1.$$

The vectors ϵ, η, $\zeta \times \xi$ and ζ, ξ, $\epsilon \times \eta$ therefore satisfy the relations for reciprocal systems. If they be chosen as α, β, γ and α', β', γ', the product takes the form

$$\pm (I - 2\alpha\alpha' - 2\beta\beta') = \mp (I - 2\gamma\gamma').$$

The geometric meaning of the results may be summed up in a theorem in which it will be convenient to put some additional but obvious statements concerning the converse problem.

THEOREM 4. *Two commutative reflections compound into the identical transformation when and only when they are identical with one another. And conversely the identical transformation may be resolved in ∞^4 ways into the product of reflections which are identical. Two commutative reflections compound into a central reflection when and only when one is planar and the other is linear with the same fundamental line and plane. Conversely the resolution may be effected in ∞^4 ways into such reflections. Two commutative reflections compound into a planar reflection when and only when they are either central and linear or planar and linear; in the former case the line and plane remain the same, in the latter case the plane of the resultant reflection is the plane of the two lines of the component reflections and the line of the resultant reflection is the line of intersection of the planes of the components. Conversely the resolution of a planar reflection into two commutative reflections may be accomplished in ∞^2 ways by choosing the lines of the desired component reflections at random in the plane of the given reflection and taking the planes of the components through these lines and the line of the given reflection. Two commutative reflections compound into a linear reflection when and only when they are either central and planar or both planar or both linear; in the first case the line and plane remain the same, in the latter cases the line is the intersection of the planes of the components and the plane is the plane through the two lines of the components. A converse similar to the above holds good.*

The analysis has shown that the product of two commutative reflections is a reflection and has furnished specific information as to what reflection the product is. If it had been a matter of no importance to determine the resultant reflection, the fact that it was a reflection could have been inferred as a special case of the general theorem: If the product of two involutory transformations

is commutative, the product is itself involutory. This general theorem follows from the symbolic equations

$$t_1 t_2 = t_2 t_1 = t_2^{-1} t_1^{-1} = (t_1 t_2)^{-1},$$

where t_1 and t_2 are involutory. A similar set of equations,

$$t_1 t_2 = (t_1 t_2)^{-1} = t_2^{-1} t_1^{-1} = t_2 t_1,$$

shows that conversely: If the product of two involutory transformations is involutory, the product is commutative. This enables us to omit the word commutative from the above theorem. Hence

THEOREM 5. *The word commutative may be deleted from theorem* 4*; for the compositions and resolutions there indicated exist when and only when the component reflections are commutative, that is, when and only when they satisfy the conditions of theorem* 3.

By *definition* two reflections, planar or linear, will be said to be *complanar* when their fundamental planes coincide, and *collinear* when their fundamental lines coincide, irrespective of whether the reflections belong to the same type or not. It has been seen that two reflections which are both complanar and collinear compound into the identical transformation or into the central reflection according as they are of the same or of different types. The next question is concerning the composition of reflection which are either complanar or collinear, but not both.

The product of two complanar planar reflections may be written as

$$(I - 2\epsilon\zeta)\cdot(I - 2\eta\zeta), \qquad \epsilon\cdot\zeta = \eta\cdot\zeta = 1.$$

This is immediately reducible to

$$I + 2(\epsilon - \eta)\zeta.$$

The effect of applying this dyadic to any vector ρ is

$$\rho' = \rho + 2(\epsilon - \eta)\zeta\cdot\rho.$$

The condition $\epsilon\cdot\zeta = \eta\cdot\zeta$ gives $(\epsilon - \eta)\cdot\zeta = 0$. Hence $\epsilon - \eta$ is perpendicular to ζ, or, in other words, lies in the plane represented by ζ. Hence the transformation consists in adding to any vector a component in the definite direction $\epsilon - \eta$ and of magnitude proportional to the component of that vector perpendicular to the plane represented by ζ. Points in the plane represented by ζ are unchanged in position.

Now by definition a special simple shear* is a transformation which leaves a plane fixed point for point and which moves points not in this plane along lines

* *Vector Analysis*, p. 363. Note that in this case and those which follow, the coefficient of I is 1. This is necessitated by the fact that for any transformation compounded of reflections there is no dilatation.

parallel to a given direction in the plane by an amount proportional to their distance above the plane. Thus a special simple shear is completely specified when its fixed plane, a direction in the plane, and a number which gives the amount of motion in that direction per unit distance from the plane is given. This amount is easy to determine in the above case. Analytically it is $2(\epsilon - \eta)$: for ζ may be taken as a unit vector perpendicular to the common plane, and then, since ρ must be taken of unit length in the direction of ζ if the amount of shift at unit distance from the plane be desired, we have $\xi \cdot \rho = 1$. Now the relations $\epsilon \cdot \zeta = \eta \cdot \zeta = 1$ show that ϵ and η terminate in a plane at unit distance above the fixed plane. Hence

THEOREM 6. *The product of two complanar planar reflections is a special simple shear parallel to the common plane and in the direction of the intersection of this plane with the plane of the two lines of the reflections and of amount equal (per unit distance from the fixed plane) to twice the distance from the point where the line of the first* reflection cuts a plane at a unit distance from the fixed plane to the point where the line of the second* reflection cuts this plane.*†

Two remarks will be sufficient to lead to two more theorems. In the first place the identity

$$(I - 2\epsilon\zeta)\cdot(I - 2\eta\zeta) = (2\epsilon\zeta - I)\cdot(2\eta\zeta - I)$$

shows that no new work need be done on complanar linear reflections. In the second place, it should be noted that any simple shear of the type above considered may be written in the form

$$I + \gamma\beta, \qquad \gamma \cdot \beta = 0.$$

Now if two reflections be chosen with a common plane perpendicular to β and if the choice of the lines ϵ and η be made to conform to the relations

$$\gamma = 2(\epsilon - \eta), \qquad \epsilon \cdot \beta = \eta \cdot \beta = 1,$$

it is evident that the shear has been resolved into two complanar planar reflections. Moreover the choice of the direction of either ϵ or η is arbitrary — the choice of either one confines the other to a perfectly definite direction in the plane of the one first chosen and of γ. Hence

THEOREM 7. *The product of two complanar linear reflections is a special simple shear parallel to the common plane in the direction of the intersection of this*

* It should be noted that the *first* and *second* reflections are respectively $I - 2\eta\zeta$ and $I - 2\epsilon\zeta$, that is, the order of the reflections is necessarily the opposite of that in which the product of the dyadics is written. The reason for this is obvious.

† The value of this amount may be expressed trigonometrically by direct translation from the vector formulas. If $I - 2\epsilon\zeta$ be written as $I - 2\epsilon_1\zeta_1 \sec(\epsilon\zeta)$, where ϵ_1 and ζ_1 are unit vectors, the evaluation of the amount of shear gives

$$\sqrt{\sec^2(\epsilon\zeta) + \sec^2(\eta\zeta) - 2\sec(\epsilon\zeta)\sec(\eta\zeta)\cos(\epsilon\eta)}.$$

plane with the plane of the two lines of the reflections and of amount equal (per unit distance from the fixed plane) to twice the distance from the point where the line of the first reflection cuts a plane at unit distance from the fixed plane to the point where the line of the second reflection cuts this plane.

THEOREM 8. *A special simple shear may be resolved in ∞^2 ways into the product of two planar or two linear reflections which are complanar with the fixed plane of the shear, and the line of one reflection may be chosen at random.* The line of the other reflection will then be uniquely determined by the conditions stated in theorems* 6 *and* 7.

The discussion may be carried on in similar manner for the case of collinear reflections. The product

$$(I-2\epsilon\zeta)\cdot(I-2\epsilon\xi) \qquad \epsilon\cdot\zeta=\epsilon\cdot\xi=1,$$

reduces to the form

$$I+2\epsilon(\xi-\zeta).$$

The vector ϵ is unchanged because of the relations

$$I\cdot\epsilon=\epsilon, \qquad \xi\cdot\epsilon-\zeta\cdot\epsilon=0.$$

The vector $\xi\times\zeta$ is likewise unchanged because of the relations

$$I\cdot\xi\times\zeta=\xi\times\zeta, \qquad \xi\cdot\xi\times\zeta=\zeta\cdot\xi\times\zeta=0.$$

Hence points in the plane of ϵ and $\xi\times\zeta$ remain fixed. This is the plane of the shear which the product obviously represents. The result of applying the shear to any vector ρ is

$$\rho'=I\cdot\rho+2\epsilon(\xi-\zeta)\cdot\rho=\rho+2\epsilon(\xi-\zeta)\cdot\rho.$$

Hence the direction of the shear is ϵ. The vector which is perpendicular to the plane of the shear is $\epsilon\times(\xi\times\zeta)$ or $\xi-\zeta$. The relations $\epsilon\cdot\zeta=\epsilon\cdot\xi=1$ enable one to read off the amount of the shear as twice the distance between the points where the vectors ξ and ζ meet a plane at unit distance from the origin and perpendicular to ϵ. It is, however, more convenient to note that

$$\sigma=\frac{\xi-\zeta}{|\xi-\zeta|}-\frac{(\xi-\zeta)\cdot\xi}{|\xi-\zeta|}\epsilon+y\xi\times\zeta$$

and

$$\tau=\frac{\xi-\zeta}{|\xi-\zeta|}-\frac{(\xi-\zeta)\cdot\zeta}{|\xi-\zeta|}\epsilon+y\xi\times\zeta$$

* The only exception to the at-random-ness of the choice of this line is that it shall not lie in the plane of the shear. It has been stated previously, however, that this plane is the plane of the reflections, and hence the line could not lie in this plane. Frequently this same state of affairs occurs. We shall therefore make the convention that at random, applied to the choice of a line or plane of a reflection, means at random except for such positions as make the line and plane of the reflection occupy this position of coincidence, which would render the reflection meaningless.

are two vectors terminating in a plane at unit distance from the fixed plane of the shear, that they lie respectively in the planes represented by ξ and ζ, and that their difference is precisely the magnitude and direction of the shear. Hence, adding a few obvious propositions, we may state

THEOREM 9. *The product of two collinear reflections, be they both planar or both linear, is a special simple shear parallel to the plane that contains their common line and the line of intersection of their planes and in the direction of their common line and of amount equal (per unit distance from the fixed plane) to twice the distance from the point where a line parallel to and at a unit distance above the fixed plane cuts the plane of the first reflection, to the point where it cuts the plane of the second reflection. And conversely any such shear may be resolved in ∞^2 ways into the product of two collinear reflections; for the common line may be chosen as the direction of the shear, and the line of intersection of the planes may be taken as any line in the plane of the shear, and the two planes of the reflections may be so set as to cut off one half the amount of the shear upon a line parallel to the line of the shear at unit distance from the plane of the shear.*

If two complanar or collinear reflections one of which is planar and the other linear be compounded the result is of the form

$$-(I-2\epsilon\zeta)\cdot(I-2\eta\zeta) \qquad \text{or} \qquad -(I-2\epsilon\zeta)\cdot(I-2\epsilon\xi).$$

Apart from the initial negative sign this is merely a shear, as before. The negative sign reverses all the directions in space. A transformation which is in all respects like a shear except that all directions in space have been reversed may be called a *perverted** shear. Then

THEOREM 10. *The relation of two complanar or collinear reflections, of which one is planar and the other linear, to the perverted shear is the same as the relation of two like complanar or collinear reflections to the (unperverted) shear. See theorems* 6–9.

There have been found a number of ways in which a special simple shear may be resolved into the product of two reflections. The question arises whether all the possibilities have been enumerated. The corresponding question in the case of resolving a reflection into two reflections was answered in theorem 5 by means of general equations in the theory of involutory transformations. This method is not available here. Collineations may, however, be classified, in the simpler cases, as essentially (projectively) different according as they have similar or dissimilar characteristic equations. This method has been carried out in detail for strains in the *Vector Analysis*, pages 356–367. The facts are in part these: Any dyadic Φ satisfies identically a definite cubic equation †

* Analogously to the nomenclature of GIBBS. *Vector Analysis*, p. 337.

† In general if ϕ denote a dyadic representing a strain in n dimensions and if $\phi_s, \phi_{2s}, \cdots, \phi_n$ be its invariants, the dyadic satisfies identically the equation of the nth order

$$\phi^n - \phi_s\phi^{n-1} + \phi_{2s}\phi^{n-2}\cdots + (-1)^n\phi_n I = (\phi - a_1 I)\cdot(\phi - a_2 I)\cdot\cdots\cdot(\phi - a_n I) = 0$$

$$\Phi^3 - \Phi_S \Phi^2 + \Phi_{2S}\Phi - \Phi_3 = 0$$

or

$$(\Phi - aI)\cdot(\Phi - bI)\cdot(\Phi - cI) = 0,$$

where the quantities a, b, c are roots of the cubic equation

$$x^3 - \Phi_S x^2 + \Phi_{2S} x - \Phi_3 = 0.$$

If two or three roots of this equation are equal the dyadic may satisfy an equation of lower degree. The equation of lowest degree which the dyadic satisfies is called the *characteristic* equation. There are six distinct types of characteristic equation for strains* (if no distinction is made between real and imaginary roots of the scalar cubic). To test a given strain it is merely necessary to find what type of characteristic equation it has.

In the case of the special simple shear the characteristic equation is of the second order and takes the form

$$(\Phi - I)^2 = 0.$$

The roots of the cubic are all equal to unity (for we are considering merely the case where there is no stretching modulus), and the cubic is

$$x^3 - 3x^2 + 3x - 1 = 0.$$

For the perverted shear the roots are all equal to -1 instead of $+1$ and the characteristic equation becomes

$$(\Phi + I)^2 = 0,$$

while the scalar cubic is

$$x^3 + 3x^2 + 3x + 1 = 0.$$

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where $a_1, a_2, \cdots, a_n$ are the roots of a scalar equation of the nth degree. To classify the strain it is convenient to classify first all different sorts of sets of roots which the scalar n-ic can have, according to their multiplicity, and then to subclassify according to the form of the characteristic equation. Thus in the case $n=4$ the different types are: a, b, c, d; a, b, c, c; a, a, c, c; a, a, a, b; a, a, a, a. In the first case the characteristic equation is determined as a product of four factors: $\phi - aI$, $\phi - bI$, $\phi - cI$, $\phi - dI$. Not so, however, in the other cases. In the second case, the factor $\phi - cI$ may occur twice, leaving the equation of the nth degree, or it may occur only once, reducing the degree the $(n-1)$st. Similarly in the third case there may be four factors to the characteristic equation, or one of the factors may drop out, or two of them may drop out, reducing the equation to $(\phi - aI)\cdot(\phi - cI)$. There are thus three types of strain corresponding to this set of roots for the scalar n-ic. If all the types be counted up, it appears that there are 13 types of characteristic equation at most. That each of these types exists, is easily shown by actually writing down the expression for it. This classification is not complete except when $n=2$ or 3; for larger values of n additional criteria must be adduced. Thus when $n=4$, there are 14 types of strain, but only 13 types of characteristic equations, and when $n=5$ these numbers are 27 and 24 respectively. This difficulty need not concern us here.

* *Vector Analysis*, p. 366. The first two types there given differ only by the fact that in one case two of the roots are conjugate imaginaries. Geometrically this is an essential difference analytically it is unimportant.

This case we will not take up in detail. The entire treatment resembles too closely that for the (unperverted) shear.

Consider any reflection

$$\Phi = \pm (I - 2\epsilon\zeta), \quad \Phi_2 = 2\zeta\epsilon - I, \quad \Phi_S = +1, \quad \Phi_{2S} = -1, \quad \Phi_3 = +1.$$

The invariants Φ_S, Φ_{2S}, Φ_3 which are the coefficients of the scalar cubic may be computed for a product of dyadics by means of the formulas*

$$(\Psi \cdot \Omega)_2 = \Psi_2 \cdot \Omega_2, \qquad (\Phi \cdot \Omega)_3 = \Phi_3 \Omega_3.$$

Hence the scalar cubic for the product

$$\Phi = \pm (I - 2\epsilon\zeta) \cdot (I - 2\eta\xi) = \pm (I - 2\epsilon\zeta - 2\eta\xi + 4\zeta \cdot \eta\epsilon\xi)\dagger$$

has the form

$$x^3 \mp (4\zeta \cdot \eta\epsilon \cdot \xi - 1)x^2 + (4\epsilon \cdot \xi\zeta \cdot \eta - 1)x \mp 1 = 0.$$

If this product is to be a shear of the type we have been considering, it is necessary that

$$\Phi_S = 3 = \pm (4\zeta \cdot \eta\epsilon \cdot \xi - 1), \qquad \Phi_{2S} = 3 = 4\epsilon \cdot \xi\zeta \cdot \eta - 1, \qquad \Phi_3 = 1 = \pm 1.$$

The first or last of these relations shows that the two reflections must be of the same type. That is, the upper sign must hold throughout. Since the upper sign holds, form the difference

$$\Phi - I = 2(\epsilon\zeta + \eta\xi - 2\epsilon\xi\zeta \cdot \eta).$$

To multiply out the expression $(\Psi - I)^2$ and hence obtain the condition for a simple shear would be long—a shorter method is desirable. Now if the product $\Psi \cdot \Omega$ of two dyadics is zero, either Ψ or Ω must be linear. Hence in this case $\Phi - I$ is linear. For this the necessary and sufficient condition is that $(\Phi - I)_2$ vanish.‡ But

$$(\Phi - I)_2 = 4\epsilon \times \eta\zeta \times \xi = 0.$$

Hence either ϵ and η are collinear or ζ and ξ are collinear; that is, the reflections are either collinear or complanar. It has already been seen that they cannot be complanar and collinear. Hence

THEOREM 11. *The necessary and sufficient condition that a special simple shear be resoluble into two reflections is that the reflections be of the same type, either linear or planar, and that they be complanar or collinear, but not both. The resolution may then be accomplished in ∞^2 ways as specified in theorems 6–9; similar results hold for the perverted shear.*

* *Vector Analysis*, p. 312.

† Here and throughout the following work the upper signs belong together and the lower signs belong likewise together, forming only two possible sets.

‡ *Vector Analysis*, p. 315.

Another case where the roots of the scalar cubic are all equal to unity is that of the complex shear of which the characteristic equation is

$$(\Phi - I)^3 = 0.$$

This shear may be reduced to the canonical form *

$$I + \alpha\beta' + \beta\gamma'.$$

The geometric properties of the transformation are these: There is a direction, the direction α, which is unchanged. Vectors parallel to α do not change either in magnitude or in direction. Through the line there is a plane, the plane of α and β, in which points move parallel to α by an amount proportional to their distance from the fixed line α. Through the line there is a second plane, the plane of α and γ, the points of which move parallel to β by an amount proportional to their distance from the line α. It should be noted, however, that the points of any plane passing through the line α, say the plane of α and δ, move parallel to a certain direction in the plane of α and β, namely the direction

$$\alpha\beta' \cdot \delta + \beta\gamma' \cdot \delta.$$

Thus the reduction of the complex shear to the canonical form may be accomplished in a single infinity of ways. A complex strain may therefore be specified by giving its fixed line a fixed plane through this line and a number which expresses the shift parallel to the line (per unit distance from the line), and a second plane through the fixed line with the direction and amount of shift (per unit distance from the line) of the points is this plane.

In distinction from the complex shear, there is the perverted complex shear, which is the combination of the shear with reversal of all directions in space. The characteristic equation of the perverted complex shear is

$$(\Phi + I)^3 = 0$$

and its scalar cubic is

$$x^3 + 3x^2 + 3x + 1 = 0.$$

This case is so similar to that of the (unperverted) shear that it need not be taken up in detail.

Consider then the product of two reflections. Just as before, it appears that the reflections must be of the same type, either planar or linear, if their product is to be a complex shear. The upper sign therefore holds. The value of the invariant Φ_{2S} is

$$\Phi_{2S} = 3 = 4\epsilon \cdot \xi\zeta \cdot \eta - 1.$$

* *Vector Analysis*, pp. 365–7.

Hence

$$\zeta\cdot\eta\epsilon\cdot\xi = 1.$$

But the relations

$$\epsilon\cdot\zeta = \eta\cdot\xi = 1 \qquad \text{and} \qquad \epsilon\times\eta\cdot\zeta\times\xi = \epsilon\cdot\zeta\eta\cdot\xi - \eta\cdot\xi\epsilon\cdot\xi$$

show that the condition reduces to

$$\epsilon\times\eta\cdot\zeta\times\xi = 0.$$

And in this equation neither $\epsilon\times\eta$ nor $\zeta\times\xi$ can vanish, or the product would reduce to the case of a simple shear previously considered. Hence

THEOREM 12. *The necessary and sufficient condition that the product of two reflections be a complex shear is that the reflections be of the same type and that the lines of the two reflections be complanar with the intersection of the planes of the reflections without the reflections themselves being either complanar or collinear. An analogous result holds for perverted complex shears, the only difference being that the reflections must be of different type.*

Further to discuss the transformation write the product in the form

$$\Phi = I - 2(\eta - \zeta\cdot\eta\epsilon)\xi - 2\epsilon(\zeta - \zeta\cdot\eta\xi).$$

Here

$$(\eta - \zeta\cdot\eta\epsilon)\cdot\xi = \eta\cdot\xi - \zeta\cdot\eta\epsilon\cdot\xi = 0,$$

$$\epsilon\cdot(\zeta - \zeta\cdot\eta\xi) = \epsilon\cdot\zeta - \zeta\cdot\eta\epsilon\cdot\xi = 0,$$

$$(\eta - \zeta\cdot\eta\epsilon)\cdot(\zeta - \zeta\cdot\eta\xi) = 0.$$

Hence the vectors $\eta - \zeta\cdot\eta\epsilon$, ϵ, ξ, $\zeta - \zeta\cdot\eta\xi$ satisfy the relations which are characteristic of the vectors α, β, β', γ' that enter into the canonical expression for the complex shear (with the exception of the relation $\beta\cdot\beta' = 1$ that depends on the magnitudes of the vectors). The vector $\eta - \zeta\cdot\eta\epsilon$ which corresponds to α is the fixed direction of the shear. By inspection, however, the vector $\zeta\times\xi$ is unchanged—as is geometrically obvious. And in fact it is evident, on expanding, that

$$(\eta - \zeta\cdot\eta\epsilon)\times(\zeta\times\xi) = 0.$$

The direction ϵ, which is the line of the second reflection, is the other fundamental line in the fixed plane. In particular a vector $a\zeta\times\xi + b\epsilon$ becomes

$$a\zeta\times\xi + b\epsilon + 2b\left(\epsilon - \frac{1}{\zeta\cdot\eta}\eta\right).$$

Hence the amount of motion of the points in the fixed plane is

$$2b\left(\epsilon - \frac{1}{\zeta\cdot\eta}\eta\right).$$

Moreover, since $\epsilon\cdot\zeta = 1$, ϵ may be regarded as terminating in the plane determined by ζ, and if η be regarded as terminating in this same plane, $\eta\cdot\zeta = 1$. Then the additional term which represents the motion in the fixed plane reduces to $2b(\epsilon - \eta)$, with the interpretation that the points move parallel to the fixed line $\zeta \times \xi$ by an amount equal (per unit distance from that line) to twice the distance from the point where the first line η cuts a line at a unit distance from that line to the point where the second line ϵ cuts the same line.

The expression for the product shows that any vector ρ in the plane of $\zeta \times \xi$ and ξ suffers no change parallel to the fixed line $\zeta \times \xi$ but is altered by the amount $-2\epsilon\zeta\cdot\rho$ in the direction ϵ. By virtue of the relation $\epsilon\cdot\zeta = 1$ the equation

$$-\epsilon\zeta\cdot\rho = (\epsilon \times \rho) \times \zeta - \rho$$

is identical and shows that the change $-2\epsilon\zeta\cdot\rho$ is twice the distance from the extremity of ρ to the point where a line parallel to ϵ and passing through this extremity cuts the plane represented by ζ. To sum up we have

THEOREM 13. *The product of two planar or two linear reflections which are neither complanar nor collinear but so related that their two lines and the line of intersection of their two planes are complanar gives rise to a complex shear of which the fixed line is the line of intersection of the two planes, and of which the fixed plane is the common plane of the lines* ϵ, η, $\zeta \times \xi$. *The amount of the shift parallel to the fixed line and in the fixed plane is equal (per unit distance from the fixed line) to twice the distance from the intersection of the first line* η *to the intersection of the second line* ϵ *with a line lying in the fixed plane at a unit distance from the fixed line. The plane in which there is no shift parallel to the fixed line is the plane of the first reflection, and points in this plane are shifted parallel to* ϵ *by an amount* equal (per unit distance from the fixed line) to twice the distance from a point in the plane of the first reflection and at a unit distance from the fixed line to the point in the plane of the second reflection where a line through this point and parallel to* ϵ *cuts the plane of the second reflection. An analogous result for the perverted complex shear.*

THEOREM 14. *Conversely any complex shear may be resolved into the product of two planar or two linear reflections by choosing the lines of the two reflections in the fixed plane of the shear and subject to the sole restriction that the distance of the*

* The deduction of the trigonometric relations that express the magnitude of the shear is left to the reader.

points where the first and second line respectively cut the line at a unit distance from the fixed line of the shear shall be equal to one-half the amount of shift (per unit distance from the fixed line) in the fixed plane; and by choosing the two planes of the reflections to pass through the fixed line of the shear in such a way that the first plane coincides with the plane whose points are shifted parallel to the line of the second reflection and that the second plane be distant from the first by an amount measured along a line parallel to the second line and through a point at a unit distance from the fixed line equal to one-half the amount of the shift due to the shear. This resolution may therefore be accomplished in ∞^1 ways and not in ∞^2 ways.

The case in which the lines of the reflections lay in the planes, each of the other, gave rise to commutative reflections which compound into another reflection. If the scalar cubic and the characteristic equation had been called into play, it would have appeared that this case corresponds to the roots $-1, -1, +1$ or $-1, +1, +1$ and to the characteristic equations

$$(\Phi - I)\cdot(\Phi + I) = 0 \qquad \text{or} \qquad (\Phi + I)\cdot(\Phi + I) = 0,$$

according as the resulting reflection was linear or planar. There arises then the question: What if only one of the lines lie in the plane of the other reflection? In this case either $\zeta\cdot\eta = 0$ or $\xi\cdot\epsilon = 0$; but not both. There is no need of giving the details of the computation. The result shows that the characteristic equation is of the third order and takes the form

$$(\Phi - I)\cdot(\Phi + I)^2 = 0 \qquad \text{or} \qquad (\Phi + I)\cdot(\Phi - I)^2 = 0,$$

according as the roots are the first or second of the above set.

This sort of equation betokens a simple shear* (non-special). The general shear of this type is

$$\alpha\alpha' - p(\beta\beta' + \gamma\gamma') - \gamma\beta' \qquad \text{or} \qquad \alpha\alpha' + p(\beta\beta' + \gamma\gamma') + \gamma\beta'.$$

Here, in the first case, there is a fixed direction α. Through this direction pass two planes, of which one, the plane of α and γ, is fixed as a whole, though if vectors are resolved parallel to α and γ the γ-components are stretched in the ratio $-p : 1$ and the α-components in the ratio $a : 1$. The points of the other plane, that of α and β, have a compound motion. The vectors which denote the points are stretched along α and β and take on an increment parallel to γ of amount proportional to their components along β. The plane of β and γ is stretched in the ratio $p : 1$ and the points move out of it by an amount proportional to their β components, the motion being in the direction γ. The vector β may be chosen at will in the plane of β and γ. In the case of the above roots the shear takes the form

* *Vector Analysis*, pp. 366-7.

$$\alpha\alpha' - (\beta\beta' + \gamma\gamma') - \gamma\beta' \qquad \text{or} \qquad -\alpha\alpha' + (\beta\beta' + \gamma\gamma') + \gamma\beta'.$$

The latter is merely a perversion of the first.

If the line of the first reflection lies in the plane of the second, $\zeta \cdot \eta = 0$, and the product takes the form (we shall consider, as usual, the first or unperverted case)

$$\Phi = I - 2\epsilon\zeta - 2\eta\xi.$$

It is obvious that the line $\zeta \times \xi$ of intersection of the two planes plays the rôle of the fixed line α; the line η, namely that which lies in the plane of other reflection, plays the rôle of γ, parallel to which the shear takes place; the plane of the two lines ϵ and η, that of the plane of β and γ. The amount of the shear suffered by ϵ is

$$-2\eta\xi \cdot \epsilon, \qquad \text{but} \qquad (\epsilon - \xi \cdot \epsilon\eta) \cdot \xi = 0;$$

which shows that this amount is twice the distance from the plane of ϵ and α to the plane ξ, if that distance be measured parallel to η.

If it is the line of the second reflection that lies in the plane of the first, the product becomes

$$\Phi = I - 2\epsilon\zeta - 2\eta\xi + 4\zeta \cdot \eta\epsilon\xi, \qquad \epsilon \cdot \xi = 0.$$

The line of intersection $\zeta \times \xi$ of the two planes still corresponds to the fixed line α. The line ϵ is now the line parallel to which the shear takes place. Thus in either case it is the line which lies in the plane of the other reflection that determines the direction of the shear. The plane corresponding to the plane of β and γ is still the plane of the two lines. The amount of shear suffered by η is

$$2\epsilon\zeta \cdot \eta, \qquad \text{but} \qquad (\eta - \epsilon\zeta \cdot \eta)\zeta = 0;$$

which shows that this amount* is twice the distance from the plane ζ to the plane of η and α, if the distance be measured parallel to ϵ.

Conversely any simple shear all of whose roots are numerically equal to unity (but not all equal algebraically) may be resolved in ∞^1 ways into the product of two reflections. This may be accomplished by choosing the planes of the reflections through the fixed line and one of them coincident with the fixed plane. The line of the other reflection must then be chosen in this plane and along the direction of the shear parallel to the plane. The line of the other reflection may then be taken at will in a definite plane, after which the final plane is wholly determined by the magnitude of the shear — but differently according as

* The trigonometric values are again left to the reader.

it is the line of the first reflection which is to lie in the plane of the second or vice versa. It will not be necessary to specify all these relations in a theorem or to prove, as may readily be done, that these are the only possible resolutions. It will be sufficient to state

THEOREM 15. *The necessary and sufficient condition that two reflections compound into a simple (non-special) shear of which the roots are numerically equal to unity is that the line of one of the reflections lie in the plane of the other. And conversely any such shear may be resolved into two reflections in ∞^1 ways.*

2. *The general case of the composition of two or three reflections.*

The general product of two reflections and the scalar cubic associated therewith have been given on page 282. The form of the cubic leads to

THEOREM 16. *If a dyadic Φ is the product of two roots of the idemfactor the conditions*

$$\Phi_{2S} = \pm \Phi_S, \qquad \Phi_3 = \pm 1$$

hold. The scalar cubic is a reciprocal equation with one of the three roots equal to ± 1 and with the product of the other two roots equal to $+1$.

As one of the roots of the cubic is ± 1, the equation may be written

$$(x \mp 1)[x^2 \mp (\Phi_{2S} - 1)x + 1] = 0.$$

The condition that the roots of the second factor be equal is

$$\Phi_{2S} - 1 = \pm 2, \qquad \Phi_{2S} = 3 \text{ or } -1.$$

The three roots are then some combination selected from ± 1, ± 1, ± 1. Geometrically these cases correspond to the identical transformation, the reflections, and the two shears—all of which have been treated in detail in § 1. If the coefficient of the first degree term in the quadratic factor be numerically decreased from the value which gives the equal roots, the roots of the resulting quadratic will be imaginary, whereas if that coefficient be increased the roots will be real. Hence the inequalities

$$|\Phi_{2S} - 1| > 2, \quad \Phi_{2S} > 3 \quad \text{or} \quad \Phi_{2S} < -1, \quad \epsilon \cdot \xi\zeta \cdot \eta < 0 \quad \text{or} \quad \epsilon \cdot \xi\zeta \cdot \eta > 1$$

obtain when two roots of the cubic are real, and the dyadic reduces to a tonic; while the inequalities

$$|\Phi_{2S} - 1| < 2, \qquad -1 < \Phi_{2S} < 3, \qquad 0 < \epsilon \cdot \xi\zeta \cdot \eta < 1$$

hold when two roots are imaginary and the dyadic reduces to the cyclotonic.

Just as in the case of shears, it will be necessary to distinguish between perverted and unperverted tonics and cyclotonics. The same convention will be adopted, namely, that if the value of Φ_3 is $+1$ so that one of the roots of the cubic is $+1$ the transformation is unperverted, but when Φ_3 is -1 so that one root of the cubic is -1 the transformation will be called perverted. In any transformation which is the product of two reflections there is one line which either remains unchanged both in magnitude and in direction or is merely reversed in direction without change of magnitude. This is the line of intersection of the two planes; and in the former case the transformation is unperverted, while in the latter case it is perverted.

Let the attention be hereafter confined to the unperverted types. The computation of the second of the product gives

$$\Phi_2 = I - 2\zeta\epsilon - 2\xi\eta + 4\xi\cdot\epsilon\zeta\eta.$$

This shows that areas in the plane denoted by $\epsilon \times \eta$, that is, in the plane of the two lines ϵ and η are unchanged. Now the entire result of the transformation may be expressed in terms of the transformation along $\zeta \times \xi$ which is the identical transformation and the transformation in the plane of ϵ and η. Thus it becomes possible to read off the general results in the present case from those obtained previously.* They may be stated as

THEOREM 17. *The product of two planar or two linear reflections which relative to one another have none of the special relations discussed in* § 1, *is a transformation which leaves the line of intersection of the two planes unchanged whether in magnitude or in direction and which leaves area in the plane of the two lines invariant. The fixed lines of the transformation in this plane divide harmonically the two pairs of lines formed by associating the line of each reflection with the line in which its plane cuts this fixed plane. The transformation is tonic or cyclotonic according as its fixed lines are real or imaginary, that is, according as the two pairs of lines do not or do separate each other. The result for the product of two reflections of which one is planar and the other linear is the same except that the transformation is perverted. And conversely any tonic or cyclotonic which leaves one line invariant in magnitude and direction and which has an invariant plane not passing through this line and subject to the invariancy of areas may be resolved into the product of two reflections both planar or both linear by choosing the invariant line as the intersection of the two planes and by arranging the lines of the reflections and the lines in which their planes cut the fixed plane so as to give the required*

* If the statements of the following theorem 17 are not entirely clear from the very brief reasoning given here, a reference to my *A generalized conception of area: applications to collineations in the plane*, Annals of Mathematics, ser. 2, vol. 5 (1903), pp. 29–45, will doubtless make them evident. It seems undesirable to increase the dimensions of the present discussion by practically repeating portions of this earlier work.

transformation in this plane. As this may be done in ∞^1 ways, the above resolution may be accomplished in ∞^1 ways. A similar result holds for perverted tonics and cyclotonics.*

In the general classification of strains there are seven types if distinction between reals and imaginatives is made. In the classification of unimodular strains there are these seven types, but the additional relation $\Phi_3 = \pm 1$ limiting the product of the roots of the scalar cubic to ± 1 modifies the result somewhat. If the roots are all alike and if the transformation is real (which has been constantly the point of view of this discussion) the roots are either plus or minus one. This gives the identical transformation or the central reflection (the perverted identical transformation) if the characteristic equation is $\Phi \mp I = 0$; the simple special shear or its perversion, if the characteristic equation is $(\Phi \mp I)^2 = 0$; the complex shear or its perversion, if the equation is $(\Phi \mp I)^3 = 0$. If two of the roots are alike, they must be $+1, +1, -1$ or $+1, -1, -1$ or $a, a, 1/a^2$. The first two cases give the planar and linear reflections respectively, if the characteristic equation is $(\Phi \pm I)\cdot(\Phi \mp I) = 0$; the simple shear, if the equation is

$$(\Phi + I)\cdot(\Phi - I)^2 = 0 \qquad \text{or} \qquad (\Phi - I)\cdot(\Phi + I)^2 = 0.$$

Finally if the roots are $\pm 1, k, 1/k$ the transformation is a tonic or cyclotonic resoluble into two reflections, and if the roots are not of this type the result is a tonic or cyclotonic not so resoluble.

It should be noted that theorem 16 states that if a transformation is resoluble into two reflections

$$\Phi_{2s} = \mp \Phi_s, \qquad \Phi_3 = \mp 1,$$

according as it is perverted or unperverted. Now conversely if these relations hold, the cubic becomes a reciprocal equation and the roots cannot be of the form

$$a, a, \pm 1/a^2 \qquad \text{or} \qquad a, b, c \text{ (not } \pm 1, k, 1/k).$$

That is, the roots must bring the transformation under one of the several heads which has already been found to be resoluble into two reflections. Hence theorem 16 may be completed in

THEOREM 18. *The necessary and sufficient condition that a strain be resoluble into two reflections is that the invariants of the strain satisfy the relations*

$$\Phi_{2s} = \pm \Phi_s, \qquad \Phi_3 = \pm 1,$$

that is, the scalar cubic shall be reciprocal.

* It is interesting to note the fact, which might be stated as a theorem, that if a strain is resoluble into two reflections, it is so resoluble in ∞^1, ∞^2, or ∞^4 ways according as its characteristic equation is of the 3rd, the 2nd, or the 1st order.

To return to the geometry relative to the cyclotonic and tonic. The cyclotonic of which the characteristic equation is

$$(\Phi - aI)\cdot(\Phi^2 - 2p\cos q\Phi + p^2 I) = 0$$

may be put in the form *

$$a\alpha\alpha' + p\cos q(\beta\beta' + \gamma\gamma') + p\sin q(\gamma\beta' - \beta\gamma').$$

The scalar cubic takes the form

$$(x - a)(x^2 - 2p\cos q\cdot x + p^2) = 0.$$

Now it has been shown that if the transformation is the product of two reflections the scalar cubic may be written as

$$(x \mp 1)[x^2 \mp (\Phi_{2S} - 1)x + 1] = 0.$$

Comparing corresponding coefficients shows that in this case the cyclotonic (confining the attention to the unperverted type) reduces to

$$p=1,\ \alpha\alpha' + \cos\cos^{-1}[(\Phi_{2S}-1)/2](\beta\beta' + \gamma\gamma') + \sin\cos^{-1}[(\Phi_{2S}-1)/2](\gamma\beta' - \beta\gamma').$$

This is a *cyclic* dyadic and represents the transformation which has been called an elliptic rotation through the angle $\cos^{-1}[(\Phi_{2S} - 1)/2]$. If $m = 2\pi/\cos^{-1}[(\Phi_{2S} - 1)/2]$ this dyadic may be regarded as the mth root of the idemfactor.† Hence

THEOREM 19. *The necessary and sufficient condition that a cyclotonic be resoluble into the product of two reflections is that it be merely cyclic. The angle of the elliptic rotation is* $\cos^{-1}[(\Phi_{2S} - 1)/2] = \cos^{-1}(2\zeta\cdot\eta\xi\cdot\epsilon - 1)$.

THEOREM 20. *Any root* ‡ *of the idemfactor may be written in* ∞^1 *ways as the product of two square roots.*

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* *Vector Analysis*, pp. 355, 361, 366.

† *Vector Analysis*, pp. 348–350.

‡ It has not been shown that even if the cyclic dyadic is a root of the idemfactor, every root must conversely be of this type. To do this proceed as follows: First the determinant of the idemfactor is unity, and the determinant of a pth root of the factor must therefore be a pth root of unity. If the strain is to be real, this pth root must be ± 1 (we may consider p as a prime, if we desire, and then the root would become $+1$). Next the equation $(\sqrt[p]{I})_2 = (\sqrt[p]{I})_3(\sqrt[p]{I})_c^{-1}$ shows that the values of $(\sqrt[p]{I})_{2S}$ and $(\sqrt[p]{I})_S$ are such as to make the scalar cubic reciprocal. Now if the roots of this equation are all real they must be all equal to 1, and there must be a complete set of fixed directions. Hence the root of the idemfactor reduces to $\pm I$, according as p is odd or even. But if two of the roots of the scalar cubic are conjugate imaginaries the pth root of I takes on cyclic form either unperverted or perverted. Thus it is proved that every root of the idemfactor is of this type.

It is not difficult to write down the ellipse in which the rotation takes place. It will be more convenient, as the rest of the work is in space, to write down the cylinder which has this ellipse for director curve and the line of intersection of the two planes for generator. The three directions ϵ, $\zeta \times \xi$, $(\epsilon \times \eta) \times \zeta$, of which the last represents the line of intersection of the ζ-plane with the plane of ϵ and η, are three conjugate directions in the quadric surface.* The three directions reciprocal to these are ζ, $\epsilon \times (\zeta \times \xi)$, $\epsilon \times \eta$ and the quadric may therefore be written in the form †

$$a\zeta\zeta + b\epsilon \times (\zeta \times \xi)\epsilon \times (\zeta \times \xi) + c\epsilon \times \eta\epsilon \times \eta.$$

The condition that the quadric be a cylinder parallel to $\zeta \times \zeta$ necessitates the vanishing of c, and the fact that η and $\xi \times (\epsilon \times \eta)$ are conjugate directions gives

$$\eta \cdot [a\zeta\zeta + b\epsilon \times (\zeta \times \xi)\epsilon \times \cdot (\zeta \times \xi)] \cdot \xi \times (\epsilon \times \eta) = 0.$$

If the ratio of a to b is determined from this condition, the final form of the cylinder turns out to be

$$\rho \cdot [(1 - \epsilon \cdot \xi\zeta \cdot \eta)\epsilon \cdot \xi\zeta\zeta + \eta \cdot \zeta\epsilon \times (\zeta \times \xi)\epsilon \times (\zeta \times \xi)] \cdot \rho = \text{const.}$$

The above deduction of the equation of the conic in which the points of planes parallel to the plane of the two lines ϵ and η move is in no wise restricted to the case of elliptic rotation. If the transformation were a tonic which leaves area invariant the motion of the points of the plane would be in hyperbolas, and the above equation would be that of the hyperbolic cylinders of which the director curves are these hyperbolas and of which the elements are parallel to $\zeta \times \xi$. In fact the form of the equation shows that the type is elliptic or hyperbolic according as

$$(1 - \epsilon \cdot \xi\zeta \cdot \eta)\epsilon \cdot \zeta \qquad \text{and} \qquad \zeta \cdot \eta$$

have the same or opposite signs — which is equivalent to the criterion established to distinguish between the cyclotonic and tonic. This suggests the question of whether it may not be possible to explain the transformation in case it is a tonic as a sort of hyperbolic rotation, and thus introduce a greater symmetry into the treatment of the tonic and cyclotonic.

It should be remembered that in elliptic rotation the dyadic

$$\Phi = \alpha\alpha' + \cos q(\beta\beta' + \gamma\gamma') + \sin q(\gamma\beta' - \beta\gamma')$$

used as a prefactor advances a radius vector in an ellipse of which β and γ are

* This follows from the harmonic property stated in theorem 17 and from the harmonic properties of conjugate directions.

† *Vector Analysis*, p. 378.

conjugate diameters through a sector, of which the area is to the area of the whole ellipse as q is to 2π.* In order to extend this statement to the case of the hyperbola it is necessary to replace the area of the ellipse by the area of the rectangle constructed on the semi-axes of the ellipse. Then the radius vector advances through a sector, the area of which is to the area of the rectangle as q is to 2. From analogy we state

THEOREM 21. *A dyadic of the form*

$$\Phi = \alpha\alpha' + \cosh q(\beta\beta' + \gamma\gamma') + \sinh q(\gamma\beta' + \beta\gamma')$$

used as a prefactor advances a radius vector in a hyperbola of which the vectors β and γ are conjugate semi-diameters through a sector, of which the area is to the area of the rectangle constructed on the semi-axes (or the parallelogram on any conjugate semi-diameters) as q is to 2.

As the hyperbola of which β and γ are conjugate radius vectors may be projected into a rectangular hyperbola in such a manner that the said conjugate direction become the axes of the rectangular hyperbola and all areas remain the same, and as the statement of the problem admits a transformation of similitude with the origin as center, it will be sufficient to prove the theorem in the case of the hyperbola $x^2 - y^2 = 1$ and the dyadic

$$\mathbf{ii} + \cosh q(\mathbf{jj} + \mathbf{kk}) + \sinh q(\mathbf{kj} + \mathbf{jk}).$$

Any radius vector in the hyperbola may be written as

$$\rho = \mathbf{i}\cosh p + \mathbf{j}\sinh p$$

and then

$$\rho = \Phi\cdot\rho = \mathbf{i}\cosh(p+q) + \mathbf{j}\sinh(p+q).$$

The area of the sector of the hyperbola from the horizontal to any inclination is

$$\tfrac{1}{2}\int_0^\theta \rho\cdot\rho d\theta = \tfrac{1}{2}\int_0^p (\cosh^2 p + \sinh^2 p)\, d\tan^{-1}\tanh p = \tfrac{1}{2}p.$$

Hence the theorem is proved. It might be noted that this tonic has the analytic form of a $2\pi/(q\sqrt{-1})$th root of the idemfactor.

The criterion for a versor† is $\Phi\cdot\Phi_C = I$, $\Phi_3 = 1$. To apply this test to ascertain under what conditions the product of two reflections is a versor would be less simple than to proceed from the point of view of the elliptic rotation. If the ellipse becomes a circle the rotation becomes versorial, and conversely.

* *Vector Analysis*, p. 349.

† *Vector Analysis*, p. 335.

Hence for the versor $\epsilon, \eta, \zeta, \xi$ lie in a plane perpendicular to the axis of version. Moreover as the conjugate diameters of a circle are perpendicular to each other, the lines of each reflection must be normal to its plane of the reflection. There are similar results for the perversor. Hence

THEOREM 22. *The necessary and sufficient condition that the product of two reflections be a versor is that the reflections be orthogonal and of the same type; if one is planar and the other linear, the result is a perversor.*

The actual expression for the elliptic or hyperbolic rotation has not yet been given in canonical form in terms of the elements which enter into the reflections of which the transformation is compounded. There is no difficulty in doing this. In fact the vectors which correspond to the α, β, γ of the canonical form may be chosen as $\zeta \times \xi$, ϵ, $(\epsilon \times \eta) \times \zeta$. If m denote the quantity $\xi \cdot \epsilon\zeta \cdot \eta$ the reciprocals are

$$\alpha' = \frac{\epsilon \times \eta}{m-1}, \qquad \beta' = \zeta, \qquad \gamma' = \frac{\epsilon \times (\zeta \times \xi)}{m-1}.$$

From these the canonical form may be written down whether for the elliptic or hyperbolic type by following the models on pages 291–3. The fixed lines of the transformation which are the asymptotes of the ellipse or hyperbola are along the directions

$$\eta + [\sqrt{(m-1)m} \pm m]\,\epsilon.$$

We pass on to the case of the resolution of the collineation or strain with a fixed point and a unit modulus into three reflections. Let Ω represent the strain. If the modulus is $\Omega_3 = +1$, all three of the reflections are linear or two of them are planar and the third is linear; whereas if the modulus is $\Omega_3 = -1$, two are linear and one is planar or all three are planar. Suppose the case to be that of the first of these four. It is required to find a reflection $2\epsilon\zeta - I$ such that

$$[\Omega \cdot (2\epsilon\zeta - I)]_{2S} = [\Omega \cdot (2\epsilon\zeta - I)]_S.$$

By use of the relation $(\Omega \cdot \Phi)_2 = \Omega_2 \cdot \Phi_2$ this reduces to

$$2\epsilon \cdot \Omega_2 \cdot \zeta - \Omega_{2S} = 2\epsilon \cdot \Omega_C \cdot \zeta - \Omega_S.$$

But we have always $\Omega_3 I = \Omega_C \cdot \Omega_2$, and hence $\Omega_2 = \Omega_C^{-1}$. That is to say,

$$2\zeta \cdot (\Omega - \Omega^{-1}) \cdot \epsilon = \Omega_S - \Omega_S^{-1}, \qquad \Omega_S^{-1} = (\Omega^{-1})_S \neq (\Omega_S)^{-1},$$

is the equation which must be satisfied by ϵ and ζ.

The other three possible cases may be reduced to these. For in the second case Ω_3 is still equal to $+1$, but the first reflection applied may be planar.

Then the product $\Omega\cdot(I-2\epsilon\zeta)$ would have a negative modulus and the condition that it be resoluble into two reflections would be

$$[\Omega\cdot(I-2\epsilon\zeta)]_{2S}=-[\Omega\cdot(I-2\epsilon\zeta)]_S,$$

which is precisely the same as before. If on the other hand $\Omega_3=-1$, we may effect the resolution of $-\Omega$ as above. But the reciprocal of $-\Omega$ is $-\Omega^{-1}$. Thus the condition is unchanged. The form of the condition may be changed by noting that

$$\zeta\cdot\epsilon=\zeta\cdot I\cdot\epsilon=1 \qquad \text{and} \qquad \Omega_S-\Omega_S^{-1}=\zeta\cdot(\Omega_S-\Omega_S^{-1})I\cdot\epsilon$$

and hence

$$\zeta\cdot[(\Omega-\Omega^{-1})-\tfrac{1}{2}(\Omega_S-\Omega_S^{-1})I]\cdot\epsilon=0.$$

We may therefore state

THEOREM 23. *In order that ϵ and ζ may serve as line and plane, respectively, for the first of three reflections into which a strain Ω may be resolved, it is necessary and sufficient that they satisfy the equation*

$$\zeta\cdot\Psi\cdot\epsilon=0, \qquad \Psi=(\Omega-\Omega^{-1})-\tfrac{1}{2}(\Omega_S-\Omega_S^{-1})I.$$

First consider the cases in which Ω is itself resoluble into two reflections. Here

$$\Omega_{2S}=\Omega_3\Omega_{CS}^{-1}=\Omega_3\Omega_S^{-1}=\pm\Omega_S \qquad \text{and hence} \qquad \Omega_S-\Omega_S^{-1}=0,$$

and the condition reduces to

$$\zeta\cdot(\Omega-\Omega^{-1})\cdot\epsilon=0.$$

The dyadic $\Omega-\Omega^{-1}$ is then always incomplete; for

$$(\Omega-\Omega^{-1})_3=\Omega_3-\Omega_2:\Omega^{-1}+\Omega:\Omega_2^{-1}-\Omega_3^{-1}=\Omega:\Omega_2^{-1}-\Omega_2:\Omega^{-1}.$$

But

$$\Omega:\Omega_2^{-1}=(\Omega\cdot\Omega_{2C}^{-1})_S=(\Omega^2)_S \qquad \text{and} \qquad \Omega_2:\Omega^{-1}=(\Omega_2^2)_S.$$

Hence the question resolves itself into the question whether the square of a transformation resoluble into two reflections is itself so resoluble. The answer is affirmative, as may be seen by investigating the squares of the various types that have been obtained: but though this method leads to a number of interesting relations, it is too long to take up here. The general expression for any dyadic may be written* as

$$\Omega=a\alpha\alpha'+g\alpha\beta'+f\alpha\gamma'+b\beta\beta'+e\beta\gamma'+c\gamma\gamma'$$

in a triply infinite system of ways. The coefficients of the diagonal terms are

* This general form may be recognized at once by checking it against the various types we have noted as possible in the classification of strains. It is, however, merely the special case in three dimensions of a general form to which a collineation may be reduced. This form is not so

the roots of the scalar cubic. The square of this dyadic is, as far as concerns the terms in the diagonal,

$$\Omega^2 = a^2\alpha\alpha' + b^2\beta\beta' + c^2\gamma\gamma' + 3 \text{ terms.}$$

Now if one of the quantities a, b, c is numerically unity and the other two are reciprocals, the same is true of a^2, b^2, c^2. Hence

THEOREM 24. *If Ω is resoluble into the product of two reflections the dyadic $\Omega - \Omega^{-1}$ is incomplete, and the condition on ϵ and ζ becomes*

$$\zeta \cdot \Phi \cdot \epsilon = 0, \qquad \Phi = \Omega - \Omega^{-1}, \qquad \Phi_s = 0,$$

and conversely.

It is possible to write $\Omega - \Omega^{-1}$ as $\Omega^{-1} \cdot (\Omega^2 - I)$, and all the degrees of incompleteness must occur in the factor $\Omega^2 - I$. If this vanishes, Ω is itself a reflection; if it is linear, all three roots of the cubic are numerically unity and the transformation is one of the simple shears, special or non-special; if it is merely planar, the transformation is a complex shear or an elliptic or hyperbolic rotation. In the first of these three cases ϵ may be chosen at pleasure and so may ζ. The condition is always satisfied. In the second case, if ϵ is taken perpendicular to the consequent of Φ, ζ may be chosen at pleasure, whereas if ϵ is not so chosen, ζ must be selected from the vectors perpendicular to the antecedent of Φ and not perpendicular to ϵ. It would be impossible to make this choice if ϵ had been chosen parallel to the antecedent of Φ, that is, if $\Phi \cdot \epsilon = x\epsilon$ and $x \neq 0$. But in this case, as Φ is linear, $\Phi \cdot \epsilon = \epsilon\Phi_s = 0$. Hence any choice of ϵ is admissible. In the last case any choice for ζ (not perpendicular to ϵ) is permissible if ϵ be taken perpendicular to the plane of the consequents of Φ. But if ϵ be not so taken, then ζ is restricted to plane perpendicular to $(\Omega - \Omega^{-1}) \cdot \epsilon$. There may be two independent values for ϵ, for which $(\Omega - \Omega^{-1}) \cdot \epsilon$ becomes parallel to ϵ without vanishing, and the indicated resolution then becomes impossible. A brief examination of this case shows that the

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widely used as it would be if authors were less willing to assume the "general case" of a complete system of fixed points when discussing collineations. (It may be noted in passing that the group of translations does not contain a single transformation which can be said to belong to the "general case.") The general form to which any collineation may be reduced is

$$\begin{aligned}
\rho x_1' &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \\
\rho x_2' &= \quad a_{22}x_2 + \cdots + a_{2n}x_n \qquad \rho x_{n-1}' = a_{n-1\,n-1}x_{n-1} + a_{n-1\,n}x_n \\
. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad .

only directions ϵ for which $(\Omega - \Omega^{-1})\cdot\epsilon$ is parallel to ϵ without vanishing are the fixed directions of the rotations. These results and some others which it is not difficult to verify may be stated in the following theorems:

THEOREM 25. *An elliptic or hyperbolic rotation may be resolved into three reflections by taking the line of the first reflection at random (with the exception of the two fixed lines in the plane of the rotation) and taking the plane through a specified line; or by assuming the plane of the first reflection at random and choosing the line in a specified plane. There is one exceptional choice of the line and one of the plane such that the choice of the other remains arbitrary.*

THEOREM 26. *A complex shear may be resolved into three reflections by taking the line of the first reflection at random and taking the plane through a specified line, etc., as in theorem* 25.

THEOREM 27. *A simple shear of the types considered above, whether special or non-special, may be resolved into the product of three reflections by taking the line of the first reflection at random and taking the plane through a specified line; or by assuming the plane of the first reflection at random and choosing the line in a specified plane. If, exceptionally, the line be chosen in a certain plane or the plane through a certain line, the subsequent choice of the other remains arbitrary.*

In the discussion of the case where Ω is not resoluble into two reflections, the behavior of the dyadic

$$\Psi = (\Omega - \Omega^{-1}) - \tfrac{1}{2}(\Omega_s - \Omega^{-1}_s)\,I$$

is of fundamental importance. If Ω be expressed in the reduced form

$$\Omega = a\alpha\alpha' + b\beta\beta' \pm \frac{1}{ab}\gamma\gamma' + 3 \text{ terms}$$

the dyadic Ψ takes the form

$$-\Psi = \left[\tfrac{1}{2}\left(a + b \pm \frac{1}{ab} - \frac{1}{a} - \frac{1}{b} \mp ab\right) - \left(a - \frac{1}{a}\right)\right]\alpha\alpha' + [\ \ ]\beta\beta' + [\ \ ]\gamma\gamma' + 3 \text{ terms},$$

and will be complete unless one of the coefficients of the three diagonal terms vanishes. Suppose the first one to vanish. Then

$$-\left(a - \frac{1}{a}\right) + \left(b - \frac{1}{b}\right) \pm \left(\frac{1}{ab} - ab\right) = 0$$

or

$$-a^2b + b + ab^2 - a \pm 1 \mp a^2b^2 = 0.$$

This expression is always factorable into three factors chosen from among the six $(b \pm 1)(a \pm 1)(ab \pm 1) = 0$. These are just the conditions that one of the roots be numerically unity, that is, that the strain Ω be resoluble into two reflections. Hence

THEOREM 28. *The necessary and sufficient condition that* Ω *be resoluble into two reflections is that*

$$\Psi = (\Omega - \Omega^{-1}) - \tfrac{1}{2}(\Omega_s - \Omega_s^{-1})\, I$$

be incomplete.

In case Ω is not resoluble into two reflections, it is (by virtue of this theorem) impossible to find any direction for the line ϵ which shall allow the plane ζ to be arbitrary. The question arises whether there may not be directions for ϵ which shall make it impossible to find any ζ whatsoever. The case of impossibility can only arise when $\Psi \cdot \epsilon$ is parallel to ϵ. That is, in case ϵ is one of the fixed directions of Ψ. But as every direction is fixed in the transformation kI, it is sufficient to state that ϵ is a fixed direction for $\Omega - \Omega^{-1}$, and a reference to the expression of this dyadic in reduced form will show that this is equivalent to saying that ϵ is a fixed direction of Ω. Now there are only two types of strain which we have to consider and which are not resoluble into two reflections. They are where the roots are $a, a, \pm 1/a^2$, which is a simple shear or special tonic, and where the roots are a, b, c and none of them numerically unity — the tonic or cyclotonic. In the case of the tonic or cyclotonic there are just three directions of impossibility (and for the cyclotonic two of these are imaginary), for the simple shear there are two such directions of which one is the fixed direction of the shear and the other the direction parallel to which the shearing takes place; for the special tonic there is a whole plane of such directions, the plane of the equal roots a, a.

THEOREM 29. *Any unimodular strain not resoluble into two reflections is resoluble into three reflections of which the first has an arbitrary line (provided it be not collinear with any of the fixed directions of the strain) and a plane which must pass through a definite line but is otherwise unlimited except that it should not pass through the line of the reflection.*

It may be observed in general as a result of the foregoing theorems 25–29 that to resolve a strain into three reflections we have

THEOREM 30. *In all cases the line of the first reflection may be chosen at random (except along such fixed directions of the strain as correspond to stretching) and the plane may be any plane which passes through a specified line but does not contain the line of the reflection.*

The proportions which this article has attained and this natural close make it unwise to continue here and now with the more detailed relations of oblique reflections to the unimodular strains.

YALE UNIVERSITY, NEW HAVEN, CONN.,
July 25, 1906.

ON THE INTRODUCTION OF CONVERGENCE FACTORS INTO SUMMABLE SERIES AND SUMMABLE INTEGRALS*

BY

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The object of this paper is to develop certain general theorems about convergence factors. In the case of series we shall mean by convergence factors a set of functions of a parameter which, when introduced as factors of the successive terms of the series, cause a divergent series to converge, or a series which is already convergent to converge more rapidly, throughout a certain range of values of the parameter. In the case of integrals we shall mean by a convergence factor a function of the variable of integration and a new parameter which, when introduced as a factor of the integrand, causes a divergent integral to converge, or an integral which is already convergent to converge more rapidly, throughout a certain range of values of the parameter.

Although the name convergence factor is of recent origin, the subject itself, in the simple case of a convergent series, goes back to ABEL and virtually takes its rise in his well known theorem on the continuity of a power series. The successive powers of x in the terms of a power series are, in fact, convergence factors of a simple nature, though not ordinarily regarded as such.

The first attempt to extend the theory of convergence factors to general types of divergent series was made by FROBENIUS.† The class of series which he considered will be designated in this paper as summable series‡ and may be defined as follows: Given a series

$$u_0 + u_1 + u_2 + \cdots,$$

we represent by s_n the sum of its first $(n+1)$ terms and form

$$S_n = \frac{s_0 + s_1 + s_2 + \cdots + s_n}{n+1}.$$

If the limit

$$\lim_{n=\infty} S_n$$

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† Crelle's Journal, vol. 89 (1880), p. 262.

‡ Many authors use this term in a more general sense.

exists, we say that the series is summable and attach to it the value of that limit. It is a well known and easily established theorem that every convergent series is summable, and that the process of summation leads to the value of the series as ordinarily defined.

The theorem proved by FROBENIUS in the article above mentioned includes only the simple class of convergence factors first considered by ABEL, i. e., successive powers of the parameter. A more general theorem was proved by FEJÉR,* in which the convergence factors have the form $\phi(0), \phi(t), \phi(2t), \cdots$. A similar theorem concerning still more general convergence factors was stated without proof by HARDY at the close of a recent paper in the Proceedings of the London Mathematical Society.† The present investigation had been begun and the essential results of §§ 1 and 2 obtained and communicated to Professor BÔCHER before the appearance of HARDY's paper.

The principal results of the present paper are contained in Theorems I, II, IV, and V. The method of proof in these four theorems is a development of that used by RIEMANN in proving a certain lemma (cf. RIEMANN, *Mathematische Werke*, 2d ed., p. 246). This lemma appears as a special case of Theorem II if we take $f_n(\alpha) = \sin^2 n\alpha / n^2\alpha^2$.

§ 1. *Convergence factors in summable series.*

THEOREM I. *If the series*

$$u_0 + u_1 + u_2 + \cdots$$

is summable and has the value S, then the series

$$F(\alpha) = u_0 + u_1 f_1(\alpha) + u_2 f_2(\alpha) + \cdots \tag{1}$$

will be absolutely convergent and continuous for all positive values of α, and will approach S as its limit when $\alpha = +0$, provided the convergence factors $f_1(\alpha), f_2(\alpha), \cdots$ satisfy the following conditions:

$$(a) \qquad f_n(\alpha) \text{ is continuous} \qquad (\alpha \geqq 0),$$

$$(b) \qquad |f_n(\alpha)| < \frac{N}{n^{2+\rho}\alpha^{2+\rho}} \qquad (\alpha > 0),$$

$$(c) \qquad f_n(0) = 1,$$

*Mathematische Annalen, vol. 58 (1904), p. 62.

†Ser. 2, vol. 4 (1906), part 4; p. 247. HARDY'S results are somewhat similar to Theorems I and II below, but the conditions imposed on the convergence factors are decidedly different, and his theorems are on the whole less far reaching. His requirement, in the case of summable series, that the differences between two successive convergence factors be positive, will, I think, be found unnecessary if a method of proof analogous to the proof of Theorem I of this article be followed.

$$(d) \quad f_n(\alpha) - 2f_{n+1}(\alpha) + f_{n+2}(\alpha) \geqq 0^* \quad (0 \leqq \alpha \leqq c;\ 0 \leqq n\alpha \leqq c;\ n = 0, 1, \cdots),\dagger$$

$$(e) \quad |f_n(\alpha) - 2f_{n+1}(\alpha) + f_{n+2}(\alpha)| < \frac{L}{n^{2+\rho}\alpha^\rho} \quad (\alpha > 0),$$

where N, ρ, c, and L are positive constants.

We must first derive the following further condition which is satisfied by the convergence factors,

$$(f) \quad |f_n(\alpha) - f_{n+1}(\alpha)| < \frac{L_1}{n^{1+\rho}\alpha^\rho} \quad (\alpha > 0),$$

where L_1 is a positive constant.

We have from condition (b)

$$\lim_{n=\infty} [f_n(\alpha) - f_{n+1}(\alpha)] = 0 \quad (\alpha > 0),$$

and consequently

$$f_n(\alpha) - f_{n+1}(\alpha) = \sum_{n=n}^{\infty} \{ [f_n(\alpha) - f_{n+1}(\alpha)] - [f_{n+1}(\alpha) - f_{n+2}(\alpha)] \}$$
$$= \sum_{n=n}^{\infty} \{ f_n(\alpha) - 2f_{n+1}(\alpha) + f_{n+2}(\alpha) \}.$$

Hence by virtue of condition (e)

$$|f_n(\alpha) - f_{n+1}(\alpha)| < \sum_{n=n}^{\infty} \frac{L}{n^{2+\rho}\alpha^\rho} < \int_{n-1}^{\infty} \frac{L}{x^{2+\rho}\alpha^\rho}\, dx = \frac{L}{(1+\rho)(n-1)^{1+\rho}\alpha^\rho}$$

$$(2) \quad = \frac{\frac{L}{1+\rho}}{n^{1+\rho}\alpha^\rho}\left(\frac{n}{n-1}\right)^{1+\rho} < \frac{2^{1+\rho}\frac{L}{1+\rho}}{n^{1+\rho}\alpha^\rho} \quad (\alpha > 0;\ n \geqq 2).$$

For $n = 1$ we have from condition (b)

$$(3) \quad |f_1(\alpha) - f_2(\alpha)| < \frac{2N}{\alpha^{2+\rho}} \leqq \frac{2N}{\alpha^{1+\rho}} \quad (\alpha \geqq 1),$$

and from condition (a)

$$(4) \quad |f_1(\alpha) - f_2(\alpha)| < \frac{M}{\alpha^{1+\rho}} \quad (0 < \alpha < 1),$$

where M is the greatest numerical value of $[f_1(\alpha) - f_2(\alpha)]$ in the interval $0 \leqq \alpha \leqq 1$. Choosing as L_1 the greatest of the thre equantities M, $2N$, and $2^{1+\rho}L/(1+\rho)$ we get (f) by combining (2), (3), and (4).

We shall next prove that the series (1) is absolutely convergent for every positive value of α. Let

$$s_n = \sum_{n=0}^{n} u_n,$$

* We may substitute for the inequality in (d) the condition

$$(d') \quad f_n(\alpha) - 2f_{n+1}(\alpha) + f_{n+2}(\alpha) \leqq 0.$$

If (d) (or (d')) holds for all values of n and α, (e) is unnecessary.

† For $n = 0$ we take here $f_0(\alpha) = 1$.

$$S_n = \frac{s_0 + s_1 + \cdots + s_n}{n+1}.$$

If for the sake of uniformity we let $S_{-2} = S_{-1} = 0$, we have

$$u_n = (n+1)S_n - 2nS_{n-1} + (n-1)S_{n-2} \qquad (n = 0, 1, 2, \cdots), \tag{5}$$

and consequently

$$\frac{u_n}{n} = \left(1 + \frac{1}{n}\right) S_n - 2S_{n-1} + \left(1 - \frac{1}{n}\right) S_{n-2} \qquad (n = 1, 2, \cdots),$$

whence it follows that

$$\lim_{n=\infty} \left(\frac{u_n}{n}\right) = 0.$$

We can, therefore, take k so that

$$\left|\frac{u_n}{n}\right| < k \qquad (n \geqq 1).$$

Combining this with condition (b) we see that the terms of (1) are, if we omit the first, less in absolute value than the corresponding terms of the series

$$\sum_{n=1}^{\infty} \frac{Nk}{\alpha^{2+\rho} n^{1+\rho}}.$$

Since this last series is convergent, the absolute convergence of (1) and the existence of $F(\alpha)$ for values of $\alpha > 0$ follow at once.

We may by virtue of (5) write

$$F(\alpha) = \sum_{n=0}^{\infty} \{(n+1)S_n - 2nS_{n-1} + (n-1)S_{n-2}\} f_n(\alpha). \tag{6}$$

In the series (6) we have a right to leave out the parentheses, since it is convergent, since each parenthesis contains only three terms, and since by condition (b) the general term of the resulting series has the limit zero. Moreover, in this latter series we have a right to rearrange the terms so that terms involving the same S are grouped together, since in the rearrangement we do not carry any term over more than three terms. Hence

$$F(\alpha) = \sum_{n=0}^{\infty} (n+1) S_n \{f_n(\alpha) - 2f_{n+1}(\alpha) + f_{n+2}(\alpha)\}. \tag{7}$$

Placing

$$S_n = S + \epsilon_n \tag{8}$$

we have

$$\begin{aligned} F(\alpha) = {} & S \sum_{n=0}^{\infty} (n+1) \{f_n(\alpha) - 2f_{n+1}(\alpha) + f_{n+2}(\alpha)\} \\ & + \sum_{n=0}^{\infty} \epsilon_n (n+1) \{f_n(\alpha) - 2f_{n+1}(\alpha) + f_{n+2}(\alpha)\}. \end{aligned} \tag{9}$$

But

$$\sum_{n=0}^{n}(n+1)\{f_n(\alpha)-2f_{n+1}(\alpha)+f_{n+2}(\alpha)\}=1-(n+2)f_{n+1}(\alpha)+(n+1)f_{n+2}(\alpha).$$

Whence, by condition (b),

$$\sum_{n=0}^{\infty}(n+1)\{f_n(\alpha)-2f_{n+1}(\alpha)+f_{n+2}(\alpha)\}=1 \qquad (\alpha>0),$$

and (9) reduces to

$$F(\alpha)-S=\sum_{n=0}^{\infty}\epsilon_n(n+1)\{f_n(\alpha)-2f_{n+1}(\alpha)+f_{n+2}(\alpha)\}. \tag{10}$$

By (c) the series on the right hand side of (10) converges when $\alpha=0$ and has the value 0. Accordingly, if we can show that it is uniformly convergent for values of $\alpha\geqq 0$ our theorem will be proved, since by condition (a) each term is a continuous function of α.

Given a positive quantity δ, arbitrarily small, we wish to show that we can choose q so large that

$$\left|\sum_{n=\mu}^{n=\nu}\epsilon_n(n+1)\{f_n(\alpha)-2f_{n+1}(\alpha)+f_{n+2}(\alpha)\}\right|<\delta \qquad (\nu\geqq\mu\geqq q,\ \alpha\geqq 0) \tag{11}$$

Take $m>2$ and such that

$$|\epsilon_n|<\eta=\frac{\rho c^{2+\rho}\delta}{\rho c^{2+\rho}+\rho N+\rho c^2 L_1+2^{1+\rho}c^2 L} \qquad (n\geqq m). \tag{12}$$

We first consider values of α in the interval $0<\alpha<c/2$, and represent by s the greatest integer less than c/α. Let μ and ν be any two integers such that

$$m\leqq\mu\leqq\nu.$$

Three cases must be considered:

$$(A) \qquad \mu\leqq s<\nu,$$

$$(B) \qquad \nu\leqq s,$$

$$(C) \qquad \mu>s.$$

Beginning with case (A) we write

$$\sum_{n=\mu}^{n=\nu}\epsilon_n(n+1)\{f_n(\alpha)-2f_{n+1}(\alpha)+f_{n+2}(\alpha)\}=\sum_{\mu}^{s}+\sum_{s+1}^{\nu}=R_1+R_2. \tag{13}$$

By virtue of (d)* and (12) we have for R_1

$$|R_1|<\eta\sum_{0}^{s}(n+1)\{f_n(\alpha)-2f_{n+1}(\alpha)+f_{n+2}(\alpha)\} \tag{14}$$
$$=\eta\{1-f_{s+1}(\alpha)-(s+1)[f_{s+1}(\alpha)-f_{s+2}(\alpha)]\}.$$

* The substitution of (d') for (d) occasions only a slight modification of the proof.

From condition (b) we get

$$|f_{s+1}(\alpha)| < \frac{N}{[(s+1)\alpha]^{2+\rho}} \leqq \frac{N}{c^{2+\rho}}, \tag{15}$$

and from condition (f)

$$(s+1)|f_{s+1}(\alpha) - f_{s+2}(\alpha)| < \frac{L_1}{[(s+1)\alpha]^{\rho}} \leqq \frac{L_1}{c^{\rho}}. \tag{16}$$

By combining (14), (15), and (16) we obtain

$$|R_1| < \eta \left\{ 1 + \frac{N}{c^{2+\rho}} + \frac{L_1}{c^{\rho}} \right\}. \tag{17}$$

According to condition (e) we have for R_2

$$|R_2| < \eta \sum_{s+1}^{\nu} (n+1)|f_n(\alpha) - 2f_{n+1}(\alpha) + f_{n+2}(\alpha)| < \frac{\eta L}{\alpha^{\rho}} \sum_{s+1}^{\nu} \frac{(n+1)}{n^{2+\rho}}.$$

Furthermore

$$\sum_{s+1}^{\nu} \frac{(n+1)}{n^{2+\rho}} < \int_{s}^{\nu} \frac{x+1}{x^{2+\rho}}\, dx = \left[-\frac{1}{\rho x^{\rho}} - \frac{1}{(1+\rho)x^{1+\rho}} \right]_{s}^{\nu} < \left[\frac{1}{\rho s^{\rho}} + \frac{1}{(1+\rho)s^{1+\rho}} \right] < \frac{2}{\rho s^{\rho}}.$$

Hence

$$|R_2| < \eta \frac{2L}{\rho(s\alpha)^{\rho}}.$$

But

$$s \geqq \frac{c}{\alpha} - 1$$

or

$$s\alpha \geqq c - \alpha > c - \frac{c}{2} = \frac{c}{2},$$

so that we have finally for R_2,

$$|R_2| < \eta \frac{2^{1+\rho} L}{\rho c^{\rho}}. \tag{18}$$

Combining (13), (17), and (18) we get

$$\left| \sum_{n=\mu}^{n=\nu} \epsilon_n (n+1)\{f_n(\alpha) - 2f_{n+1}(\alpha) + f_{n+2}(\alpha)\} \right| < \eta \frac{\rho c^{2+\rho} + \rho N + \rho c^2 L_1 + 2^{1+\rho} c^2 L}{\rho c^{2+\rho}} = \delta.$$

We consider next case (B). We have

$$\left| \sum_{n=\mu}^{n=\nu} \epsilon_n (n+1)\{f_n(\alpha) - 2f_{n+1}(\alpha) + f_{n+2}(\alpha)\} \right| \leqq \eta \sum_{0}^{s} (n+1)\{f_n(\alpha) - 2f_{n+1}(\alpha) + f_{n+2}(\alpha)\}$$

$$= \eta\{1 - f_{s+1}(\alpha) - (s+1)[f_{s+1}(\alpha) - f_{s+2}(\alpha)]\}$$
$$\leqq \eta\{1 + |f_{s+1}(\alpha)| + (s+1)|f_{s+1}(\alpha) - f_{s+2}(\alpha)|\}$$
$$< \eta\left\{1 + \frac{N}{c^{2+\rho}} + \frac{L_1}{c^\rho}\right\} = \eta\frac{\rho c^{2+\rho} + \rho N + \rho c^2 L_1}{\rho c^{2+\rho}} < \delta.$$

For case (C) we have

$$\left|\sum_{n=\mu}^{n=\nu} \epsilon_n(n+1)\{f_n(\alpha) - 2f_{n+1}(\alpha) + f_{n+2}(\alpha)\}\right|$$
$$\leqq \eta\sum_{s+1}^{\nu}(n+1)|f_n(\alpha) - 2f_{n+1}(\alpha) + f_{n+2}(\alpha)|$$
$$< \frac{\eta L}{\alpha^\rho}\sum_{s+1}^{\nu}\frac{n+1}{n^{2+\rho}} < \eta\frac{2^{1+\rho}L}{\rho c^\rho} = \eta\frac{2^{1+\rho}c^2 L}{\rho c^{2+\rho}} < \delta.$$

Thus we have proved (11) in all three cases for $0 < \alpha < c/2$, the value of q being equal to m.

For values of $\alpha \geqq c/2$ we have from condition (e)

$$\left|\sum_{n=\mu}^{n=\nu} \epsilon_n(n+1)\{f_n(\alpha) - 2f_{n+1}(\alpha) + f_{n+2}(\alpha)\}\right|$$
$$\leqq \sum_{\mu}^{\nu}|\epsilon_n|\frac{\left(\frac{2}{c}\right)^\rho(n+1)L}{n^{2+\rho}} \leqq \left(\frac{2}{c}\right)^\rho LK\sum_{\mu}^{\nu}\frac{(n+1)}{n^{2+\rho}}$$

where K is the greatest numerical value of the quantities $\epsilon_0, \epsilon_1, \epsilon_2, \cdots$. Since the series

$$\sum_{n=1}^{\infty}\frac{n+1}{n^{2+\rho}}$$

is convergent, we can find m_1 such that

$$\left|\sum_{n=\mu}^{n=\nu} \epsilon_n(n+1)\{f_n(\alpha) - 2f_{n+1}(\alpha) + f_{n+2}(\alpha)\}\right| < \delta \qquad \left(\begin{matrix}\nu \geqq \mu \geqq m_1 \\ \alpha \geqq c/2\end{matrix}\right).$$

For $\alpha = 0$ the inequality (11) is obviously satisfied for all values of q since each $f_i(\alpha) = 1$. If, therefore, we choose for q the greater of the two quantities m and m_1, it will hold for all values of $\alpha \geqq 0$, as was to be proved.

Convergence factors of the form $\phi(n\alpha)$ are an important class, and it is interesting to note what restrictions may be placed on $\phi(x)$ in order that the conditions of Theorem I may be satisfied. We will show that if

(i) $$|\phi(x)| < \frac{N}{x^{2+\rho}} \qquad (x > 0),$$

(ii) $$\phi(0) = 1,$$

(iii) $\phi''(x)$ exists and $\phi''(x) \geqq 0$ * $(0 \leqq x \leqq c)$,

(iv) $|\phi''(x)| < \frac{L}{x^{2+\rho}}$ $(x > 0)$,

then the functions $\phi(0) = 1$, $\phi(\alpha)$, $\phi(2\alpha)$, $\cdots$ satisfy the conditions of Theorem I.

Condition (a) follows from the fact that the second derivative of $\phi(x)$ exists for all values of $x \geqq 0$. Conditions (b) and (c) follow at once from (i) and (ii).

We have (cf. MARKOFF, *Differenzenrechnung*, § 8)

(19) $\phi(n\alpha) - 2\phi[(n+1)\alpha] + \phi[(n+2)\alpha] = \alpha^2\phi''[(n+\theta)\alpha]$ $(0 < \theta < 2)$.

Hence by virtue of (iii) condition (d) will be fulfilled, if c is there replaced by $c/3$.

Finally, by (19) and (iv)

$$|\phi(n\alpha) - 2\phi[(n+1)\alpha] + \phi[(n+2)\alpha]| < \frac{L}{n^{2+\rho}\alpha^{\rho}} \qquad (\alpha > 0;\ n = 1, 2, \cdots),$$

and (e) is satisfied.

The theorem of FEJÉR mentioned in the introduction, in which the convergence factors are functions of $n\alpha$, is not, in its greatest generality, a special case of Theorem I, but becomes so if we exclude from it all functions whose first derivatives have an infinite number of maxima and minima in the neighborhood of the origin. I shall show this by proving that if we have a function $\phi(x)$ which satisfies the conditions of FEJÉR's theorem and whose first derivative has not an infinite number of maxima and minima in the neighborhood of the origin, then it will satisfy (i)–(iv).

FEJÉR's conditions are

(a) $|\phi(x)| < \frac{M}{x^{2+\rho}}$ $(x > 1)$,

(b) $|\phi''(x)| < \frac{M}{x^{2+\rho}}$ $(x > 1)$,

(c) $\phi(0) = 1$,

(d) $|\phi''(x)| < \mu$ † $(0 \leqq x \leqq 1)$,

where M, ρ and μ are positive constants.

From (d) it follows that $\phi(x)$ is continuous in the interval $0 \leqq x \leqq 1$. We can therefore choose a positive constant N greater than M and greater than the

* We may substitute for (iii)

(iii′) $\phi''(x) \leqq 0$ $(0 \leqq x \leqq c)$.

† (d) is not explicitly stated as a condition, but is used in the course of the proof.

greatest numerical value of $\phi(x)$ in the interval $0 \leqq x \leqq 1$, so that we have

$$|\phi(x)| < \frac{N}{x^{2+\rho}} \qquad (x>0),$$

and consequently (i) is satisfied. Condition (ii) is identical with (c). Since $\phi'(x)$ has not an infinite number of maxima and minima in the neighborhood of the origin, either (iii) or (iii′) must be satisfied. By taking for L the greater of the two quantities M and μ we have in view of (b) and (d)

$$|\phi''(x)| < \frac{L}{x^{2+\rho}} \qquad (x>0),$$

and (iv) is satisfied.

Thus we see that Theorem I, when applied to functions of $n\alpha$, includes all functions satisfying FEJÉR's conditions that are likely to be useful as convergence factors. In addition to this it includes an important class of functions that are not included in FEJÉR's theorem, i. e., functions that satisfy (i)–(iv) but whose second derivative becomes infinite in the neighborhood of the origin. An example of such a function is $e^{-\sqrt{x}}$.

A general theorem which includes FEJÉR's theorem as a special case but does not include functions of the type just mentioned is obtained by replacing condition (d) of Theorem I by the condition

$$|f_n(\alpha) - 2f_{n+1}(\alpha) + f_{n+2}(\alpha)| < K\alpha^2 \qquad (0 \leqq n\alpha \leqq c).$$

The proof is quite similar to the proof of Theorem I.

§ 2. *Convergence factors in convergent series.*

Theorem I is evidently applicable to convergent series since every convergent series is summable. When dealing with convergent series, however, we do not need to place as much restriction upon the convergence factors as in the general case of a summable series. This is shown by the following theorem which in many respects is more general than Theorem I:

THEOREM II. *If the series*

$$u_0 + u_1 + u_2 + \cdots \tag{20}$$

converges to the value A, then the series

$$u_0 + u_1 f_1(\alpha) + u_2 f_2(\alpha) + \cdots \tag{21}$$

will be absolutely and uniformly convergent for all values of $\alpha \geqq 0$ and therefore will define a continuous function, provided the convergence factors *

* That the series (21) converges more rapidly than (20) appears from condition (b).

$f_1(\alpha), f_2(\alpha), \cdots$ satisfy the following conditions:

(a) $\qquad f_n(\alpha)$ *is continuous* $\qquad (\alpha \geqq 0)$,

(b) $\qquad |f_n(\alpha)| < \dfrac{N}{n^{1+\rho}\alpha^{1+\rho}} \qquad (\alpha > 0)$,

(c) $\qquad f_n(0) = 1$,

(d) $\qquad f_n(\alpha) - f_{n+1}(\alpha) \geqq 0$* $\qquad (n \geqq 1,\ 0 \leqq n\alpha \leqq c)$,

(e) $\qquad |f_n(\alpha) - f_{n+1}(\alpha)| < \dfrac{L}{n^{1+\rho}\alpha^{\rho}} \qquad (\alpha > 0)$,

where N, ρ, c and L are positive constants.

We must first derive from the given conditions the following further condition:

(f) $\qquad |f_n(\alpha)| < K \qquad (\alpha \geqq 0)$

where K is a positive constant.

We know from condition (d)† that, for a given value of α less than c, the functions $f_n(\alpha)$ form a sequence that never increases as long as $n\alpha$ does not exceed c. For each value of $\alpha < c$ we pick out the greatest integer $r \leqq c/\alpha$. This r satisfies the inequality

$$\frac{c}{2} < r\alpha \leqq c. \tag{22}$$

We have furthermore

$$f_1(\alpha) \geqq f_n(\alpha) \geqq f_r(\alpha) \qquad (0 < n\alpha < c). \tag{23}$$

But by condition (a) we can find a positive constant M such that

$$|f_1(\alpha)| < M \qquad (0 < \alpha < c),$$

and from (b) and (22)

$$|f_r(\alpha)| < \frac{N}{(r\alpha)^{1+\rho}} < \frac{2^{1+\rho}N}{c^{1+\rho}},$$

so that if we take as K_1 the greater of the two quantities M and $2^{1+\rho}N/c^{1+\rho}$, we have by (23)

$$|f_n(\alpha)| < K_1 \qquad (0 < n\alpha < c).$$

For $n\alpha \geqq c$ we have from condition (b)

$$|f_n(\alpha)| < \frac{N}{(n\alpha)^{1+\rho}} \leqq \frac{N}{c^{1+\rho}} < K_1 \qquad (n\alpha \geqq c).$$

* This is the only condition that is not satisfied by the convergence factors of Theorem I. We may substitute for it the condition

(d') $\qquad f_n(\alpha) - f_{n+1}(\alpha) \leqq 0 \qquad (0 \leqq n\alpha \leqq c).$

If (d) (or (d')) holds for all values of n and α, (e) is unnecessary.

† If we use (d') instead of (d), the inequality signs in (22) are reversed, but this does not affect the reasoning.

For $\alpha = 0$ we have from condition (c)

$$f_n(0) = 1.$$

Hence if we choose K greater than either of the two quantities K_1 and 1, we have the desired condition (f). Place now

$$u_0 + u_1 + u_2 + \cdots + u_{n-1} = A + \epsilon_n$$

so that

$$\lim_{n=\infty} \epsilon_n = 0,$$

and let δ be an arbitrarily small, positive quantity. Since by condition (a) each term of (21) is a continuous function of α, our theorem will be proved if we can show that a positive integer q exists for which

$$\left| \sum_{n=\mu}^{n=\nu} u_n f_n(\alpha) \right| < \delta \qquad (\nu \geqq \mu \geqq q\,;\ \alpha \geqq 0). \tag{24}$$

Determine a value of $m > 2$ such that

$$|\epsilon_n| < \eta = \frac{\rho c^\rho \delta}{4\rho c^\rho K + 2^\rho L} \qquad (n \geqq m). \tag{25}$$

We first consider values of α in the interval $0 < \alpha < c/2$. Then

$$\sum_{n=\mu}^{n=\nu} u_n f_n(\alpha) = \sum_{\mu}^{\nu} (\epsilon_{n+1} - \epsilon_n) f_n(\alpha) \tag{26}$$

$$= -\epsilon_\mu f_\mu(\alpha) + \sum_{\mu}^{\nu} \epsilon_{n+1} [f_n(\alpha) - f_{n+1}(\alpha)] + \epsilon_{\nu+1} f_{\nu+1}(\alpha).$$

Let s be the greatest integer less than c/α, and μ and ν any two integers for which

$$m \leqq \mu \leqq \nu.$$

Three cases must be considered:

$$\mu \leqq s < \nu, \tag{A}$$

$$\nu \leqq s, \tag{B}$$

$$\mu > s. \tag{C}$$

Beginning with (A) we put

$$\sum_{n=\mu}^{n=\nu} \epsilon_{n+1} [f_n(\alpha) - f_{n+1}(\alpha)] = \sum_{\mu}^{s} + \sum_{s}^{\nu} = R_1 + R_2. \tag{27}$$

By virtue of (d),* (f), and (25) we have for R_1

$$|R_1| \leqq \sum_{\mu}^{s} |\epsilon_{n+1}|[f_n(\alpha) - f_{n+1}(\alpha)]$$

(28)

$$< \eta \sum_{\mu}^{s} [f_n(\alpha) - f_{n+1}(\alpha)] = \eta [f_\mu(\alpha) - f_{s+1}(\alpha)] < 2K\eta.$$

For R_2 we have by condition (e)

$$|R_2| \leqq \sum_{s+1}^{\nu} |\epsilon_{n+1}[f_n(\alpha) - f_{n+1}(\alpha)]|$$

$$< \frac{\eta L}{\alpha^\rho} \sum_{s+1}^{\nu} \frac{1}{n^{1+\rho}} < \frac{\eta L}{\alpha^\rho} \int_s^\nu \frac{dx}{x^{1+\rho}} = \frac{\eta L}{\alpha^\rho} \left[\frac{-1}{\rho x^\rho} \right]_s^\nu < \frac{\eta L}{\rho s^\rho \alpha^\rho}.$$

But

$$s\alpha \geqq \left(\frac{c}{\alpha} - 1 \right) \alpha = c - \alpha > c - \frac{c}{2} = \frac{c}{2},$$

and hence

(29)
$$|R_2| < \frac{\eta 2^\rho L}{\rho c^\rho}.$$

Combining (26), (27), (28), and (29) we get

$$\left| \sum_{\mu}^{\nu} u_n f_n(\alpha) \right| \leqq |\epsilon_\mu f_\mu(\alpha)| + |R_1| + |R_2| + |\epsilon_{\nu+1} f_{\nu+1}(\alpha)|$$

$$< K\eta + 2K\eta + \frac{\eta 2^\rho L}{\rho c^\rho} + K\eta = \eta \left[\frac{4\rho c^\rho K + 2^\rho L}{\rho c^\rho} \right] = \delta.$$

We consider next case (B). We have

$$\left| \sum_{\mu}^{\nu} \epsilon_{n+1} [f_n(\alpha) - f_{n+1}(\alpha)] \right|$$

$$< \eta \sum_{\mu}^{\nu} [f_n(\alpha) - f_{n+1}(\alpha)] = \eta [f_\mu(\alpha) - f_{\nu+1}(\alpha)] < 2K\eta.$$

Combining this result with (26) we get

$$\left| \sum_{\mu}^{\nu} u_n f_n(\alpha) \right| < |\epsilon_\mu f_\mu(\alpha)| + 2K\eta + |\epsilon_{\nu+1} f_{\nu+1}(\alpha)| < 4K\eta < \delta.$$

For case (C) we have

$$\left| \sum_{\mu}^{\nu} \epsilon_{n+1} [f_n(\alpha) - f_{n+1}(\alpha)] \right| < \eta \sum_{s+1}^{\nu} |f_n(\alpha) - f_{n+1}(\alpha)| < \frac{\eta L}{\alpha^\rho} \sum_{s+1}^{\nu} \frac{1}{n^{1+\rho}} < \eta \frac{2^\rho L}{\rho c^\rho},$$

which combined with (26) gives

$$\left| \sum_{\mu}^{\nu} u_n f_n(\alpha) \right| < |\epsilon_\mu f_\mu(\alpha)| + \eta \frac{2^\rho L}{\rho c^\rho} + |\epsilon_{\nu+1} f_{\nu+1}(\alpha)| < K\eta + \eta \frac{2^\rho L}{\rho c^\rho} + K\eta < \delta.$$

* The substitution of (d') for (d) occasions only a slight modification of the proof.

Hence the desired inequality (24) is satisfied in all three cases, q being equal to m.

For all values of $\alpha \geqq c/2$ we have

$$|u_n f_n(\alpha)| < \frac{BN}{n^{1+\rho}\alpha^{1+\rho}} \leqq \frac{BN2^{1+\rho}}{n^{1+\rho}c^{1+\rho}}$$

where B is the largest numerical value of the terms $u_0, u_1, u_2, \cdots$. The series $\sum 1/n^{1+\rho}$ converges, and therefore we can choose m_1 such that

$$\left|\sum_{n=\mu}^{n=\nu} u_n f_n(\alpha)\right| < \delta \qquad \left(\nu \geqq \mu \geqq m_1;\ \alpha \geqq \frac{c}{2}\right).$$

When $\alpha = 0$ the series (21) becomes the series (20), and we can so choose m_2 that

$$\left|\sum_{n=\mu}^{n=\nu} u_n\right| < \delta \qquad (\nu \geqq \mu \geqq m_2).$$

Thus, finally, if we take for q the greatest of the three values m, m_1, and m_2, we have the desired inequality (24).

The convergence factors $e^{-n\alpha}$ satisfy the conditions of Theorem II. Hence if the series

$$a_0 + a_1 r + a_2 r^2 + \cdots$$

is convergent, the series

$$a_0 + a_1 r e^{-\alpha} + a_2 r^2 e^{-2\alpha} + \cdots$$

converges uniformly in the interval $\alpha \geqq 0$. If now we set $e^{-\alpha} = x/r$, it follows that

$$a_0 + a_1 r\left(\frac{x}{r}\right) + a_2 r^2\left(\frac{x}{r}\right)^2 + \cdots$$

or

$$a_0 + a_1 x + a_2 x^2 + \cdots$$

converges uniformly in the interval $0 \leqq x \leqq r$. Hence Theorem II includes ABEL's theorem as a special case.

§ 3. *Summable integrals. General facts and lemmas.*

Let $f(x)$ be a function which is integrable in every finite interval lying in the interval $a \leqq x$. Then the integral

$$\int_a^\infty f(x)\,dx$$

shall be said to be summable if the limit

$$\lim_{x=\infty}\left[\frac{1}{x}\int_a^x\int_a^\alpha f(\beta)\,d\beta\,d\alpha\right]$$

exists; and we attach to the integral the value of that limit. The agreement of this definition with the ordinary definition of an integral, is shown by the following theorem:

THEOREM III. *Every convergent integral is summable, and the process of summation gives the value of the integral ordinarily defined.**

Put

$$\int_a^\infty f(x)dx = A$$

and

$$\int_a^x f(x)dx = A + \epsilon(x),$$

so that

$$\lim_{x=\infty} \epsilon(x) = 0.$$

Then

$$\frac{1}{x}\int_a^x \int_a^\alpha f(\beta)d\beta d\alpha = \frac{1}{x}\int_a^x [A + \epsilon(\alpha)]d\alpha = A - \frac{aA}{x} + \frac{1}{x}\int_a^x \epsilon(\alpha)d\alpha.$$

Since the second term approaches zero as x becomes infinite, our theorem will be proved if we can show that the same is true of the third term. Let δ be a positive constant as small as we please, and choose m so that

$$|\epsilon(\alpha)| < \frac{\delta}{2} \qquad (\alpha \geqq m > 0).$$

Then for values of $x \geqq m$ we have

$$\left|\frac{1}{x}\int_a^x \epsilon(\alpha)d\alpha\right| \leqq \frac{1}{x}\left|\int_a^m \epsilon(\alpha)d\alpha\right| + \frac{1}{x}\left|\int_m^x \epsilon(\alpha)d\alpha\right|.$$

The second of these terms reduces to

$$\frac{x-m}{x}|\epsilon(\alpha')| \qquad (m < \alpha' < x)$$

and is therefore less than $\delta/2$. We can also choose p so that when $x > p$ the first term is less than $\delta/2$. Accordingly, since δ is arbitrary,

$$\lim_{x=\infty}\left[\frac{1}{x}\int_a^x \epsilon(\alpha)d\alpha\right] = 0,$$

and our theorem is proved.

I shall next establish a series of lemmas that will be needed in the proof of the theorem concerning the introduction of convergence factors into summable integrals.

* The substance of this theorem is stated without proof by HARDY in an article in the Quarterly Journal of Mathematics, vol. 35 (1903–04), p. 54.

LEMMA 1. *If $f(x)$ is uniformly continuous for values of $x \geqq k > 0$, then there exists a positive constant M such that*

$$|f(x)| < Mx \qquad (x \geqq k).$$

Let us choose δ such that

$$|f(x+h) - f(x)| < 1 \qquad (|h| \leqq \delta;\ x \geqq k). \tag{30}$$

Then for every $x \geqq k$ we can find a $y \geqq 0$ such that

$$x = k + y\delta.$$

If n is the greatest integer not exceeding y, we have by (30)

$$|f(k+y\delta) - f(k)| < n + 1 \qquad (y \geqq 0),$$

and therefore

$$\frac{|f(k+y\delta)|}{k+y\delta} < \frac{|f(k)|}{k+y\delta} + \frac{n+1}{k+y\delta}.$$

For values of y in the interval $0 \leqq y < 1$ we have, since $n = 0$,

$$\frac{|f(k+y\delta)|}{k+y\delta} < \frac{|f(k)|}{k} + \frac{1}{k} = K_1.$$

For values of $y \geqq 1$, since $y \geqq n \geqq 1$,

$$\frac{|f(k+y\delta)|}{k+y\delta} < \frac{|f(k)|}{k} + \frac{n+1}{n\delta} \leqq \frac{|f(k)|}{k} + \frac{2}{\delta} = K_2.$$

If then we choose as M the larger of the two constants K_1 and K_2, we have

$$|f(k+y\delta)| < M(k+y\delta) \qquad (y \geqq 0),$$

or

$$|f(x)| < Mx \qquad (x \geqq k).$$

LEMMA 2. *If $f(x)$ is uniformly continuous for values of $x \geqq k > 0$, then $f(x)/x^\rho$ (where ρ is any positive constant) is uniformly continuous in the same interval.*

We can assume without loss of generality that $k \geqq e$, where e is the base of natural logarithms; for, if it is not, it follows that $f(x)/x^\rho$ is uniformly continuous in the closed interval $k \leqq x \leqq e$ since it is continuous in it.

From Lemma 1 it follows that there exists a positive constant M such that

$$\left|\frac{f(x)}{x}\right| < M \qquad (x \geqq k).$$

Given ϵ positive and arbitrarily small, let us choose δ less than $\epsilon/2M$ and such that

$$|f(x+h) - f(x)| < \frac{k^\rho}{2}\epsilon \qquad (x \geqq k;\ 0 < h \leqq \delta).$$

We have

$$(31)\quad \left|\frac{f(x+h)}{(x+h)^\rho}-\frac{f(x)}{x^\rho}\right| \leqq \frac{1}{(x+h)^\rho}\,|f(x+h)-f(x)| + |f(x)|\left|\frac{1}{(x+h)^\rho}-\frac{1}{x^\rho}\right|.$$

For the first term on the right hand side we have

$$(32)\quad \frac{1}{(x+h)^\rho}\,|f(x+h)-f(x)| < \frac{1}{k^\rho}\,\frac{k^\rho\epsilon}{2} = \frac{\epsilon}{2} \qquad (x \geqq k\,;\ 0<h\leqq\delta).$$

Also from the Law of the Mean

$$\frac{1}{(x+h)^\rho}-\frac{1}{x^\rho} = \frac{-\rho h}{(x+\theta h)^{\rho+1}} \qquad (0<\theta<1),$$

and hence

$$|f(x)|\left|\frac{1}{(x+h)^\rho}-\frac{1}{x^\rho}\right| < M\,\frac{x\rho h}{x^{\rho+1}} \leqq M\,\frac{\rho}{e^\rho}\,h \qquad (x\geqq k\,;\ 0<h\leqq\delta).$$

But

$$\frac{\rho}{e^\rho} < 1 \qquad (\rho>0).$$

Consequently

$$(33)\quad |f(x)|\cdot\left|\frac{1}{(x+h)^\rho}-\frac{1}{x^\rho}\right| < M\delta < \frac{\epsilon}{2} \qquad (x\geqq k\,;\ 0<h\leqq\delta).$$

Combining (31), (32), and (33) we have

$$\left|\frac{f(x+h)}{(x+h)^\rho}-\frac{f(x)}{x^\rho}\right| < \epsilon \qquad (x\geqq k\,;\ 0<h\leqq\delta),$$

and the lemma is proved.

LEMMA 3. *If $f(x)$ is uniformly continuous for values of $x \geqq k > 0$, and the integral*

$$\int_a^\infty f(x)\,dx$$

is summable, then

$$\lim_{x=\infty}\left[\frac{f(x)}{x}\right] = 0.$$

We know from Lemma 2 that $f(x)/x$ is uniformly continuous for values of $x \geqq k$. Given ϵ, let us choose a positive $\delta < k$ and such that

$$(34)\quad \left|\frac{f(x+h)}{x+h}-\frac{f(x)}{x}\right| < \frac{\epsilon}{3} \qquad (x\geqq k\,;\ 0<h\leqq 2\delta).$$

Let

$$\frac{1}{x}\int_a^x\int_a^\alpha f(\beta)\,d\beta\,d\alpha = S(x).$$

Then

$$f(x)=\frac{d^2[xS(x)]}{dx^2}. \tag{35}$$

Moreover, since $\lim_{x=\infty} S(x)$ exists, we can so determine $x_0 \geqq k$ that

$$|S(x')-S(x)|<\frac{\delta^2\epsilon}{6} \qquad (x'>x\geqq x_0). \tag{36}$$

But we have *

$$(x+2\delta)S(x+2\delta)-2(x+\delta)S(x+\delta)+xS(x)=\delta^2 f(x+\theta\delta) \quad (0<\theta<2),$$

whence by (36)

$$\delta^2|f(x+\theta\delta)|<(x+2\delta)|S(x+2\delta)-S(x+\delta)| + x|S(x+\delta)-S(x)|<\frac{2(x+\delta)\delta^2\epsilon}{6} \quad (x\geqq x_0).$$

Consequently, since $x+\delta<2x$, we get

$$\left|\frac{f(x+\theta\delta)}{x+\theta\delta}\right|<\frac{2(x+\delta)\epsilon}{(x+\theta\delta)6}<\frac{2(x+\delta)\epsilon}{x\ 6}<\frac{2\epsilon}{3} \qquad (x\geqq x_0).$$

It follows then from (34) that

$$\left|\frac{f(x)}{x}\right|<\epsilon \qquad (x\geqq x_0),$$

and therefore

$$\lim_{x=\infty}\left[\frac{f(x)}{x}\right]=0.$$

Before taking up the other lemmas, I wish to introduce the following notation. Let

$$\int f(x)dx=\psi(x)+C_1, \qquad \int\psi(x)dx=\chi(x)+C_2. \tag{37}$$

Then

$$xS(x)=\int_a^x\int_a^\alpha f(\beta)d\beta d\alpha=\int_a^x[\psi(\alpha)-\psi(a)]d\alpha = \chi(x)-x\psi(a)-\chi(a)+a\psi(a). \tag{38}$$

LEMMA 4. *If $f(x)$ is uniformly continuous for values of $x\geqq k>0$, and the integral*

$$\int_a^\infty f(x)dx$$

is summable, then

$$\lim_{x=\infty}\left[\frac{\psi(x)}{x^2}\right]=0.$$

From Lemma 3 it follows that for a given positive but arbitrarily small ϵ, we

* Cf. MARKOFF: *Differenzenrechnung*, § 8.

can choose a positive $x_0 > a$ such that

$$\left|\frac{f(x)}{x}\right| < \frac{\epsilon}{2} \qquad (x \geqq x_0).$$

We have also from (37)

$$\psi(x) - \psi(x_0) = (x - x_0) f[x_0 + \theta(x - x_0)] \qquad (0 < \theta < 1),$$

and consequently

$$\left|\frac{\psi(x)}{x^2}\right| \leqq \left|\frac{\psi(x_0)}{x^2}\right| + \frac{x - x_0}{x} \cdot \left|\frac{f[x_0 + \theta(x - x_0)]}{x_0 + \theta(x - x_0)}\right| \cdot \frac{x_0 + \theta(x - x_0)}{x}$$

$$< \left|\frac{\psi(x_0)}{x^2}\right| + \frac{\epsilon}{2} \qquad (x \geqq x_0).$$

Choose $x_1 \geqq x_0$ and such that

$$\left|\frac{\psi(x_0)}{x^2}\right| < \frac{\epsilon}{2} \qquad (x \geqq x_1).$$

Then

$$\left|\frac{\psi(x)}{x^2}\right| < \epsilon \qquad (x \geqq x_1),$$

and accordingly

$$\lim_{x=\infty}\left[\frac{\psi(x)}{x^2}\right] = 0.$$

LEMMA 5. *If we have a function $\phi(x)$ such that*

$$\lim_{x=\infty}[\phi(x)] = C, \qquad \lim_{x=\infty}[\phi''(x)] = 0$$

where C is any constant, then it follows that

$$\lim_{x=\infty}[\phi'(x)] = 0.$$

Given ϵ positive and arbitrarily small, let us choose B such that

$$(39) \qquad |\phi(x') - \phi(x)| < \frac{\epsilon}{2}, \qquad |\phi''(x)| < \frac{\epsilon}{2} \qquad (x' > x \geqq B).$$

From the Law of the Mean

$$\phi(x + 1) - \phi(x) = \phi'(x + \theta) \qquad (0 < \theta < 1),$$

and therefore

$$(40) \qquad |\phi'(x + \theta)| < \frac{\epsilon}{2} \qquad (x \geqq B).$$

Applying again the Law of the Mean, we have

$$\phi'(x + 1) - \phi'(x + \theta) = (1 - \theta)\phi''(x + \theta') \qquad (\theta < \theta' < 1),$$

from which it follows, by (39) and (40), that

$$|\phi'(x + 1)| < |\phi'(x + \theta)| + |\phi''(x + \theta')| < \epsilon \qquad (x \geqq B).$$

Consequently

$$\lim_{x=\infty}[\phi'(x)]=0.$$

LEMMA 6. *If $f(x)$ is uniformly continuous for values of $x \geqq k>0$ and the integral*

$$\int_a^\infty f(x)\,dx$$

is summable, then

$$\lim_{x=\infty}[S'(x)]=0.$$

We have from (38)

$$S''(x)=\frac{2\chi(x)}{x^3}-\frac{2\psi(x)}{x^2}+\frac{f(x)}{x}-\frac{2(\chi(a)-a\psi(a))}{x^3}. \tag{41}$$

From Lemmas 3 and 4 it follows that

$$\lim_{x=\infty}\left[\frac{f(x)}{x}\right]=0, \qquad \lim_{x=\infty}\left[\frac{\psi(x)}{x^2}\right]=0,$$

and from equation (38) that

$$\lim_{x=\infty}\left[\frac{\chi(x)}{x^3}\right]=0.$$

Hence from (41)

$$\lim_{x=\infty}[S''(x)]=0.$$

Accordingly, since by hypothesis $S(x)$ approaches a limit when $x=\infty$, we have by Lemma 5

$$\lim_{x=\infty}[S'(x)]=0.$$

LEMMA 7. *If we have a function of two variables $\phi(\alpha, x)$ such that*

(a) $\phi(\alpha, x)$ is continuous in x and in α separately in the region $R(x_2 \geqq x \geqq x_1,\ \alpha_2 \geqq \alpha \geqq \alpha_1)$,

(b) $\phi'_x(\alpha, x)$ and $\phi''_x(\alpha, x)$ exist and

$$|\phi''_x(\alpha, x)|<C$$

in the region R, then $\phi'_x(\alpha, x)$ is continuous in α in the region R.

Take any point (α_0, x_0) that lies in the region R. Choose a constant Δx such that $x_0+\Delta x$ lies in the interval $x_2 \geqq x \geqq x_1$ and

$$|\Delta x|<\frac{\epsilon}{4C}, \tag{42}$$

where ϵ is an arbitrarily small, positive constant.

Then choose η such that

$$|\phi(\alpha', x_0+\Delta x)-\phi(\alpha_0, x_0+\Delta x)|<\frac{\epsilon}{4}\Delta x \qquad (|\alpha'-\alpha_0|<\eta),* \tag{43}$$

$$|\phi(\alpha', x_0)-\phi(\alpha_0, x_0)|<\frac{\epsilon}{4}\Delta x \qquad (|\alpha-\alpha_0|<\eta). \tag{44}$$

By the Law of the Mean

$$\frac{\phi(\alpha', x_0+\Delta x)-\phi(\alpha', x_0)}{\Delta x}=\phi'_x(\alpha', x_0+\theta\Delta x) \qquad (0<\theta<1), \tag{45}$$

$$\frac{\phi(\alpha_0, x_0+\Delta x)-\phi(\alpha_0, x_0)}{\Delta x}=\phi'_x(\alpha_0, x_0+\theta'\Delta x) \qquad (0<\theta'<1). \tag{46}$$

By subtraction of (46) from (45) and with the aid of (43) and (44) we obtain

$$|\phi'_x(\alpha', x_0+\theta\Delta x)-\phi'_x(\alpha_0, x_0+\theta'\Delta x)|<\frac{\epsilon}{4}+\frac{\epsilon}{4}=\frac{\epsilon}{2} \qquad (|\alpha'-\alpha_0|<\eta). \tag{47}$$

Applying again the Law of the Mean we have

$$\phi'_x(\alpha', x_0+\theta\Delta x)-\phi'_x(\alpha', x_0)=\theta\Delta x\phi''_x(\alpha', x_0+\theta_1\theta\Delta x) \qquad (0<\theta_1<1),$$

$$\phi'_x(\alpha_0, x_0+\theta'\Delta x)-\phi'_x(\alpha_0, x_0)=\theta'\Delta x\phi''_x(\alpha_0, x_0+\theta'_1\theta'\Delta x) \qquad (0<\theta'_1<1),$$

whence it follows by subtraction that

$$|\phi'_x(\alpha', x_0)-\phi'_x(\alpha_0, x_0)|\leqq|\phi'_x(\alpha', x_0+\theta\Delta x)-\phi'_x(\alpha_0, x_0+\theta'\Delta x|$$
$$+\theta'|\Delta x||\phi''_x(\alpha_0, x_0+\theta'_1\theta'\Delta x)|+\theta|\Delta x||\phi''_x(\alpha', x_0+\theta_1\theta\Delta x)|.$$

In combination with (b), (42), and (47) this gives

$$|\phi'_x(\alpha', x_0)-\phi'_x(\alpha_0, x_0)|<\epsilon \qquad (|\alpha'-\alpha_0|<\eta),$$

and the lemma is proved.

§ 4. *Convergence factors in summable integrals.*

THEOREM IV. *If the function $f(x)$ is uniformly continuous for all values of $x\geqq a$, where a is any positive constant, and the integral*

$$\int_a^\infty f(x)dx$$

is summable and has the value S, then the integral

$$F(\alpha)=\int_a^\infty f(x)\phi(\alpha, x)dx \tag{48}$$

* It is assumed in this and in the following similar inequalities that α' lies in the interval $\alpha_2\geqq\alpha\geqq\alpha_1$.

will be absolutely convergent and continuous for all positive values of α and will approach S as its limit when $\alpha = +0$, provided the convergence factor $\phi(\alpha, x)$ satisfies the following conditions for $x \geqq a$:

(a) $\phi(\alpha, x)$ *is continuous in* (α, x) $(\alpha \geqq 0)$,

(b) $\phi_x''(\alpha, x)$ *exists and is continuous in* (α, x) $(\alpha \geqq 0)$,

(c) $$|\phi(\alpha, x)| < \frac{N}{x^{2+\rho}\alpha^{2+\rho}} \qquad (\alpha > 0),$$

(d) $$\phi(0, x) = 1,$$

(e) $$\phi_x''(\alpha, x) \geqq 0^* \qquad (0 \leqq x\alpha \leqq c),$$

(f) $$|\phi_x''(\alpha, x)| < \frac{L}{x^{2+\rho}\alpha^{\rho}} \qquad (\alpha > 0),$$

where N, ρ, c, and L are positive constants.

We must first derive from the given conditions the following further conditions which hold for $x \geqq a$:

(g) $\phi_x'(\alpha, x)$ is continuous in α $(\alpha \geqq 0)$,

(h) $$|\phi_x'(\alpha, x)| < \frac{L}{x^{1+\rho}\alpha^{\rho}} \qquad (\alpha > 0).$$

Consider the function $\phi(\alpha, x)$ in the region $R(B \geqq x \geqq a;\ \alpha \geqq 0)$. By condition (a) this function is continuous throughout R. We can also show at once that its second derivative with regard to x remains finite in the region R. For from (b) we know that a positive constant M exists such that

$$|\phi_x''(\alpha, x)| < M \qquad (B \geqq x \geqq a;\ 1 \geqq \alpha \geqq 0),$$

and from condition (f) we have

$$|\phi_x''(\alpha, x)| < \frac{L}{x^{2+\rho}\alpha^{\rho}} < \frac{L}{a^{2+\rho}} \qquad (B \geqq x \geqq a;\ \alpha > 1).$$

Accordingly the function $\phi(\alpha, x)$ satisfies the conditions of Lemma 7. Consequently $\phi_x'(\alpha, x)$ is continuous in α in the region R, and therefore in the region $x \geqq a$, $\alpha \geqq 0$.

By virtue of conditions (c) and (f) we have

$$\lim_{x=\infty} [\phi(\alpha, x)] = 0 \qquad (\alpha > 0),$$

$$\lim_{x=\infty} [\phi_x''(\alpha, x)] = 0 \qquad (\alpha > 0).$$

* We may substitute for (e) the condition

(e') $$\phi_x''(\alpha, x) \leqq 0 \qquad (0 \leqq x\alpha \leqq c).$$

If (e) (or (e')) holds for all values of x and α, (f) is unnecessary.

Hence by Lemma 5 it follows that

$$\lim_{x=\infty} [\phi'_x(\alpha, x)] = 0 \qquad (\alpha > 0), \tag{49}$$

and consequently

$$\int_x^\infty \phi''_x(\alpha, x)\,dx = -\phi'_x(\alpha, x) \qquad (\alpha > 0).$$

We have then

$$|\phi'_x(\alpha, x)| = \left|\int_x^\infty \phi''_x(\alpha, x)\,dx\right| \leqq \int_x^\infty |\phi''_x(\alpha, x)|\,dx$$

$$< \int_x^\infty \frac{L}{x^{2+\rho}\alpha^\rho}\,dx = \frac{L}{(1+\rho)x^{1+\rho}\alpha^\rho} < \frac{L}{x^{1+\rho}\alpha^\rho} \qquad (\alpha > 0).$$

Having thus established conditions (g) and (h) we can now prove the following inequalities (50) and (51) which hold when $x = a$. By condition (c) we have

$$|\phi(\alpha, a)| < \frac{N}{a^{2+\rho}\alpha^{2+\rho}} < \frac{N}{a^{2+\rho}} \qquad (\alpha > 1),$$

and by condition (a) we can find a positive constant K_1 such that

$$|\phi(\alpha, a)| < K_1 \qquad (0 \leqq \alpha \leqq 1).$$

Denoting by K the greater of the two quantities $N/a^{2+\rho}$ and K_1 we have

$$|\phi(\alpha, a)| < K \qquad (\alpha \geqq 0). \tag{50}$$

Similarly from conditions (h) and (g) we see that a positive constant C can be found such that

$$|\phi'_x(\alpha, a)| < C \qquad (\alpha \geqq 0). \tag{51}$$

We shall next prove that the integral (48) is absolutely convergent for every positive value of α. By Lemma 1 there exists a positive constant M such that in the interval $x \geqq a$ we have

$$|f(x)| < Mx.$$

Consequently

$$|f(x)\phi(\alpha, x)| < \frac{MN}{x^{1+\rho}\alpha^{2+\rho}}.$$

Since also the integral

$$\int_a^\infty \frac{dx}{x^{1+\rho}}$$

is convergent, the absolute convergence of (48) and the existence of $F(\alpha)$ for values of $\alpha > 0$ follow at once.

We have from (35)

$$F(\alpha) = \int_a^\infty f(x)\phi(\alpha, x)\,dx = \int_a^\infty [2S'(x) + xS''(x)]\,\phi(\alpha, x)\,dx.$$

Integrating twice by parts we get

$$F(\alpha)=[S(x)\phi(\alpha,x)]_a^\infty+[xS'(x)\phi(\alpha,x)]_a^\infty-[S(x)x\phi'_x(\alpha,x)]_a^\infty$$
$$-\int_a^\infty S(x)\phi'_x(\alpha,x)\,dx+\int_a^\infty S(x)\,[\phi'_x(\alpha,x)+x\phi''_x(\alpha,x)]\,dx. \tag{52}$$

Now from the definition of $S(x)$ both $S(a)$ and $S'(a)$ are zero. As x becomes infinite, $S'(x)$ approaches zero by Lemma 6 and $S(x)$ by hypothesis approaches S. From condition (e) we see further that for all positive values of α, $\phi(\alpha,x)$ approaches zero as x becomes infinite.

Combining (49) with conditions (e) and (f) we obtain next

$$\lim_{x=\infty}[x\phi(\alpha,x)]=0 \qquad (\alpha>0),$$

$$\lim_{x=\infty}[2\phi'_x(\alpha,x)+x\phi''_x(\alpha,x)]=0 \qquad (\alpha>0).$$

Hence by Lemma 5

$$\lim_{x=\infty}[x\phi'_x(\alpha,x)+\phi(\alpha,x)]=0 \qquad (\alpha>0),$$

from which it follows that

$$\lim_{x=\infty}[x\phi'_x(\alpha,x)]=0 \qquad (\alpha>0).$$

The quantities in brackets in equation (52) therefore disappear, and we have

$$F(\alpha)=-\int_a^\infty S(x)\phi'_x(\alpha,x)\,dx+\int_a^\infty S(x)[\phi'_x(\alpha,x)+x\phi''_x(\alpha,x)]\,dx. \tag{53}$$

Put

$$S(x)=S+\epsilon(x)$$

so that

$$\lim_{x=\infty}\epsilon(x)=0$$

and then substitute in (53). We obtain

$$F(\alpha)=-S\int_a^\infty\phi'_x(\alpha,x)\,dx+S\int_a^\infty[\phi'_x(\alpha,x)+x\phi''_x(\alpha,x)]\,dx$$
$$-\int_a^\infty\epsilon(x)\phi'_x(\alpha,x)\,dx+\int_a^\infty\epsilon(x)[\phi'_x(\alpha,x)+x\phi''_x(\alpha,x)]\,dx. \tag{54}$$

But

$$\int_a^\infty\phi'_x(\alpha,x)\,dx=-\phi(\alpha,a) \qquad (\alpha>0),$$

$$\int_a^\infty[\phi'_x(\alpha,x)+x\phi''_x(\alpha,x)]\,dx=[x\phi'_x(\alpha,x)]_a^\infty=-a\phi'_x(\alpha,a) \qquad (\alpha>0).$$

Consequently (54) reduces to

$$F(\alpha)=\phi(\alpha,a)\,S-a\phi'_x(\alpha,a)\,S+\int_a^\infty\epsilon(x)x\phi''_x(\alpha,x)\,dx \qquad (\alpha>0). \tag{55}$$

By condition (d)

$$\phi_x'(0, x) = 0, \qquad \phi_x''(0, x) = 0,$$

and therefore the right hand member of (55) reduces to S when $\alpha = 0$. Since, by condition (b), $\phi_x''(\alpha, x)$ is a continuous function of α for all values of $\alpha \geqq 0$, our theorem is proved if we can show further that the integral in the right hand member of (55) is uniformly convergent in the interval $\alpha \geqq 0$.

Given an arbitrarily small, positive quantity, δ, let us choose $m > a$ and such that

$$|\epsilon(x)| < \eta = \frac{\rho c^{2+\rho}\delta}{(\rho+1)c^2 L + \rho c^{2+\rho}(aC+K) + \rho N} \qquad (x \geqq m). \tag{56}$$

Consider first values of α in the interval $0 < \alpha$. Let $s = c/\alpha$ and μ and ν be any two quantities such that

$$m \leqq \mu < \nu.$$

Three cases must be considered

$$(A) \qquad \mu < s < \nu,$$

$$(B) \qquad \nu \leqq s,$$

$$(C) \qquad \mu \geqq s.$$

Beginning with (A) we write

$$\int_\mu^\nu \epsilon(x) x \phi_x''(\alpha, x)\, dx = \int_\mu^s + \int_s^\nu = R_1 + R_2. \tag{57}$$

By condition (e) we have

$$|R_1| < \eta \int_\mu^s x\phi_x''(\alpha, x)dx \leqq \eta \int_a^s x\phi_x''(\alpha, x)dx.$$

But

$$\int_a^s x\phi_x''(\alpha, x)dx = s\phi_x'(\alpha, s) - a\phi_x'(\alpha, a) - \phi(\alpha, s) + \phi(\alpha, a)$$

and consequently, in view of (h), (c), (50), and (51),

$$\left|\int_a^s x\phi_x''(\alpha, x)dx\right| \leqq |s\phi_x'(\alpha, s)| + |a\phi_x'(\alpha, a)| + |\phi(\alpha, s)| + |\phi(\alpha, a)|$$

$$< \frac{L}{s^\rho \alpha^\rho} + aC + \frac{N}{s^{2+\rho}\alpha^{2+\rho}} + K = \frac{L}{c^\rho} + aC + \frac{N}{c^{2+\rho}} + K.$$

Hence

$$|R_1| < \eta\left[\frac{L}{c^\rho} + aC + \frac{N}{c^{2+\rho}} + K\right]. \tag{58}$$

From condition (f) we have for R_2,

$$|R_2| \leqq \int_a^\nu |\epsilon(x) x \phi_x''(\alpha, x)| dx < \eta \int_a^\nu \frac{L}{x^{1+\rho}\alpha^\rho} dx$$

(59)

$$= \eta \left[-\frac{L}{\rho x^\rho \alpha^\rho} \right]_a^\nu < \eta \frac{L}{\rho c^\rho}.$$

Combining (57), (58), and (59) and using (56) we have

$$\left| \int_\mu^\nu \epsilon(x) x \phi_x''(\alpha, x) dx \right| \leqq |R_1| + |R_2|$$

$$< \eta \left[\frac{(\rho+1)c^2 L + \rho c^{2+\rho}(aC+K) + \rho N}{\rho c^{2+\rho}} \right] = \delta.$$

For case (B) we have

$$\left| \int_\mu^\nu \epsilon(x) x \phi_x''(\alpha, x) dx \right| < \eta \int_\mu^\nu x \phi_x''(\alpha, x) \leqq \eta \int_a^\nu x \phi_x''(\alpha, x) dx$$

$$< \eta \left[\frac{L}{c^\rho} + aC + \frac{N}{c^{2+\rho}} + K \right] = \eta \left[\frac{\rho c^2 L + \rho c^{2+\rho}(aC+K) + \rho N}{\rho c^{2+\rho}} \right] < \delta.$$

We have finally for case (C)

$$\left| \int_\mu^\nu \epsilon(x) x \phi_x''(\alpha, x) dx \right| \leqq \int_\mu^\nu |\epsilon(x) x \phi_x''(\alpha, x)| dx$$

$$< \eta \int_\mu^\nu \frac{L}{x^{1+\rho}\alpha^\rho} dx \leqq \eta \int_a^\nu \frac{L}{x^{1+\rho}\alpha^\rho} dx < \eta \frac{L}{\rho c^\rho} = \eta \frac{c^2 L}{\rho c^{2+\rho}} < \delta.$$

Hence in all three cases

$$\left| \int_\mu^\nu \epsilon(x) x \phi_x''(\alpha, x) dx \right| < \delta \qquad \left(\nu > \mu \geqq m;\ 0 < \alpha < \frac{c}{m} \right). \tag{60}$$

For $\alpha = 0$ we have

$$\int_\mu^\nu \epsilon(x) x \phi_x''(0, x) dx = \int_\mu^\nu \epsilon(x) [0] dx < \delta \qquad (\mu \geqq a).$$

Finally, by taking for q the greater of the two quantities a and m, and by combining the results just obtained we have

$$\left| \int_\mu^\nu \epsilon(x) x \phi_x''(\alpha, x) dx \right| < \delta \qquad (\nu > \mu \geqq q;\ \alpha \geqq 0),$$

and our theorem is proved.

We assumed in Theorem IV, for the sake of simplicity of proof, that q was greater than zero. That this assumption really places no restriction upon the generality of the theorem is shown by the following corollary:

COROLLARY. *If, in Theorem* IV, *a is any constant, and the convergence factor* $\phi(\alpha, x)$ *satisfies conditions* (a), (b), (c), (d), (e), *and* (f) *for* $x \geqq b$, *where* b *is any positive constant, and conditions* (a) *and* (d) *for* $a \leqq x \leqq b$, *the theorem holds without change.*

Let

$$\int_a^b f(x)\,dx = A.$$

We have

$$\frac{1}{x}\int_a^x\int_a^\alpha f(\beta)\,d\beta d\alpha = \frac{1}{x}\int_a^x\int_b^\alpha f(\beta)\,d\beta d\alpha + \frac{1}{x}\int_a^x\int_a^b f(\beta)\,d\beta d\alpha$$

$$= \frac{1}{x}\int_b^x\int_b^\alpha f(\beta)\,d\beta d\alpha + \frac{1}{x}\int_a^b\int_b^\alpha f(\beta)\,d\beta d\alpha + \frac{1}{x}\int_a^x A\,d\alpha$$

$$= \frac{1}{x}\int_b^x\int_b^\alpha f(\beta)\,d\beta d\alpha + \frac{1}{x}\int_a^b [\psi(\alpha)-\psi(b)]\,d\alpha + \frac{x-a}{x}A,$$

and consequently

$$S = \lim_{x=\infty}\frac{1}{x}\int_a^x\int_a^\alpha f(\beta)\,d\beta d\alpha = \lim_{x=\infty}\frac{1}{x}\int_b^x\int_b^\alpha f(\beta)\,d\beta d\alpha + A$$

or

$$\lim_{x=\infty}\frac{1}{x}\int_b^x\int_b^\alpha f(\beta)\,d\beta d\alpha = S - A.$$

Hence the integral

$$\int_b^\infty f(x)\,dx$$

is summable and has the value $(S-A)$. Therefore by Theorem IV the integral

$$\int_b^\infty f(x)\phi(\alpha, x)\,dx$$

is absolutely convergent for all positive values of α, its value $G(\alpha)$ is continuous for such values, and

$$\lim_{\alpha=+0} G(\alpha) = S - A.$$

Hence

$$(61)\qquad \int_a^b f(x)\phi(\alpha, x)\,dx + \int_b^\infty f(x)\phi(\alpha, x)\,dx = \int_a^\infty f(x)\phi(\alpha, x)\,dx$$

is absolutely convergent for all positive values of α.

Let

$$\int_a^b f(x)\phi(\alpha, x)\,dx = H(\alpha).$$

Then

$$H(0) = \int_a^b f(x)\phi(0, x)\,dx = \int_a^b f(x)\,dx = A.$$

We have by (61)

$$F(\alpha) = \int_a^\infty f(x)\phi(\alpha, x)dx = H(\alpha) + G(\alpha),$$

and since $H(\alpha)$ and $G(\alpha)$ are continuous for all values of $\alpha > 0$, $F(\alpha)$ is continuous for such values. Moreover, since $H(\alpha)$ and $G(\alpha)$ each approach a limit when $\alpha = +0$, so also does $F(\alpha)$ and we have for that limit

$$\lim_{\alpha=+0} F(\alpha) = H(0) + \lim_{\alpha=+0} G(\alpha) = S.$$

§ 5. *Convergence factors in convergent integrals.*

Theorem IV is applicable to convergent integrals since by Theorem III every convergent integral is summable. As in the case of series, however, less restriction can be placed upon the convergence factor, as is shown in the following theorem:

THEOREM V. *If the function $f(x)$ is finite and integrable in every finite interval lying in the interval $x \geqq a$, where a is any positive constant, and the integral*

$$\int_a^\infty f(x)dx \tag{62}$$

converges to the value A, then the integral

$$\int_a^\infty f(x)\phi(\alpha, x)dx \tag{63}$$

will be absolutely and uniformly convergent for all values of $\alpha \geqq 0$ and therefore will define a continuous function, provided the convergence factor $\phi(\alpha, x)$ satisfies the following conditions for $x \geqq a$:*

(*a*) $\phi(\alpha, x)$ *is continuous in* (α, x) $\qquad (\alpha \geqq 0)$,

(*b*) $\phi'_x(\alpha, x)$ *exists* $\qquad (\alpha \geqq 0)$,

(*c*) $$|\phi(\alpha, x)| < \frac{N}{x^{1+\rho}\alpha^{1+\rho}} \qquad (\alpha > 0),$$

* That the integral (63) converges more rapidly than (62) can be shown by a method precisely analogous to that suggested for series in a previous footnote (see page 307) since for a given value of α, an m can be determined such that

$$\left|\int_n^\infty f(x)\phi(\alpha, x)dx\right| < \left|\int_n^\infty f(x)dx\right| \qquad (n \geqq m).$$

$$(d) \qquad \phi(0, x) = 1,$$

$$(e) \qquad \phi'_x(\alpha, x) \leqq 0 * \qquad (0 \leqq x\alpha \leqq c),$$

$$(f) \qquad |\phi'_x(\alpha, x)| < \frac{L}{x^{1+\rho}\alpha^{\rho}} \qquad (\alpha > 0),$$

where N, ρ, c, and L are positive constants.

We must first derive from the given conditions the following further condition which holds for $x \geqq a$:

$$(g) \qquad |\phi(\alpha, x)| < K \qquad (\alpha \geqq 0),$$

where K is a positive constant.

We know from condition (e) † that $\phi(\alpha, x)$, for a given value of α, either decreases or remains constant with increasing x as long as $x\alpha$ does not exceed c. We have then

$$(64) \qquad \phi(\alpha, a) \geqq \phi(\alpha, x) \geqq \phi\left(\alpha, \frac{c}{\alpha}\right) \qquad (0 < x\alpha < c).$$

But by condition (a) we can find a positive constant M such that

$$|\phi(\alpha, a)| < M \qquad (0 < \alpha < c/a),$$

and from condition (c)

$$\left|\phi\left(\alpha, \frac{c}{\alpha}\right)\right| < \frac{N}{c^{1+\rho}},$$

so that if we take as K_1 the greater of the two quantities M and $N/c^{1+\rho}$ we have by (64)

$$(65) \qquad |\phi(\alpha, x)| < K_1 \qquad (0 < x\alpha < c).$$

For $x\alpha \geqq c$ we have from condition (c)

$$(66) \qquad |\phi(\alpha, x)| \leqq \frac{N}{c^{1+\rho}} \leqq K_1 \qquad (x\alpha \geqq c).$$

For $\alpha = 0$ we have from condition (d)

$$(67) \qquad \phi(0, x) = 1.$$

* This is the only condition that is not satisfied by the convergence factor of Theorem IV. We may substitute for it the condition

$$(e') \qquad \phi'_x(\alpha, x) \geqq 0 \qquad (0 \leqq x\alpha \leqq c).$$

If (e) (or (e')) holds for all values of x and α, (f) is unnecessary.

† If we substitute (e') for (e), the inequality signs in (64) are reversed but this does not affect the reasoning.

Hence if we choose K greater than either of the quantities K_1 and 1, we have from (65), (66), and (67)

$$\phi(\alpha, x) < K \qquad (\alpha \geqq 0).$$

Let δ be an arbitrarily small positive quantity. Since $\phi(\alpha, x)$ by condition (a) is a continuous function of α for all values of $\alpha \geqq 0$, our theorem will be proved if we can show that a positive quantity q exists such that

$$\left|\int_\mu^\nu f(x)\phi(\alpha, x)\,dx\right| < \delta \qquad (\nu > \mu \geqq q;\ \alpha \geqq 0). \tag{68}$$

Determine $m > a$ and such that

$$\left|\int_\mu^\nu f(x)\,dx\right| < \eta = \frac{\rho c^\rho \delta}{L + 3K\rho c^\rho} \qquad (\nu > \mu \geqq m). \tag{69}$$

We first consider values of $\alpha > 0$. Integrating by parts we get

$$\begin{aligned}\int_\mu^\nu f(x)\phi(\alpha, x)dx &= \left[\phi(\alpha, x)\int_\mu^x f(x)\,dx\right]_\mu^\nu - \int_\mu^\nu\left[\int_\mu^x f(x)\,dx\right]\phi_x'(\alpha, x)\,dx \\ &= \phi(\alpha, \nu)\int_\mu^\nu f(x)\,dx - \int_\mu^\nu\left[\int_\mu^x f(x)\,dx\right]\phi_x'(\alpha, x)\,dx.\end{aligned} \tag{70}$$

Let $s = c/\alpha$ and let μ and ν be any two quantities such that

$$m \leqq \mu < \nu.$$

Three cases must be considered:

$$\mu < s < \nu, \tag{A}$$

$$\nu \leqq s, \tag{B}$$

$$\mu \geqq s. \tag{C}$$

Beginning with (A) we write

$$\int_\mu^\nu\left[\int_\mu^x f(x)\,dx\right]\phi_x'(\alpha, x)\,dx = \int_\mu^s + \int_s^\nu = R_1 + R_2. \tag{71}$$

By virtue of (69) and conditions (e) and (g) we have for R_1

$$\begin{aligned}|R_1| &\leqq \int_\mu^s\left|\left[\int_\mu^x f(x)\,dx\right]\phi_x'(\alpha, x)\right|dx < \eta\int_\mu^s[-\phi_x'(\alpha, x)]\,dx \\ &= \eta[\phi(\alpha, \mu) - \phi(\alpha, s)] \leqq \eta|\phi(\alpha, \mu)| + \eta|\phi(\alpha, s)| < 2K\eta \\ &\qquad \left(\nu > \mu \geqq m;\ 0 < \alpha < \frac{c}{m}\right).\end{aligned} \tag{72}$$

From condition (f) we have for R_2

$$|R_2| \leqq \int_a^\nu \left| \left[\int_\mu^x f(x)\,dx \right] \phi'_x(\alpha, x) \right| dx < \eta \int_a^\nu \frac{L}{x^{1+\rho}\alpha^\rho}\,dx$$

(73)

$$= \frac{\eta L}{\alpha^\rho}\left[-\frac{1}{\rho x^\rho} \right]_a^\nu < \frac{\eta L}{\rho(s\alpha)^\rho} = \frac{\eta L}{\rho c^\rho}.$$

From (70)–(73) we get

$$\left| \int_\mu^\nu f(x)\phi(\alpha, x)\,dx \right| \leqq |\phi(\alpha, \nu) \int_\mu^\nu f(x)\,dx| + |R_1| + |R_2|$$

$$< K\eta + 2K\eta + \frac{\eta L}{\rho c^\rho} = \eta \frac{L + 3K\rho c^\rho}{\rho c^\rho} = \delta.$$

For case (B) we have

$$\left| \int_\mu^\nu \left[\int_\mu^x f(x)\,dx \right] \phi'_x(\alpha, x)\,dx \right| \leqq \int_\mu^\nu \left| \left[\int_\mu^x f(x)\,dx \right] \phi'_x(\alpha, x) \right| dx$$

$$< \eta \int_\mu^s [-\phi'_x(\alpha, x)]\,dx = \eta[\phi(\alpha, \mu) - \phi(\alpha, s)] < 2K\eta,$$

which combined with (70) gives

$$\left| \int_\mu^\nu f(x)\phi(\alpha, x)\,dx \right| < |\phi(\alpha, \nu) \int_\mu^\nu f(x)\,dx| + 2K\eta < 3K\eta = \eta \frac{3K\rho c^\rho}{\rho c^\rho} < \delta.$$

Finally we have for case (C)

$$\left| \int_\mu^\nu \left[\int_\mu^x f(x)\,dx \right] \phi'_x(\alpha, x)\,dx \right| \leqq \int_\mu^\nu \left| \left[\int_\mu^x f(x)\,dx \right] \phi'_x(\alpha, x) \right| dx$$

$$< \eta \int_\mu^\nu |\phi'_x(\alpha, x)|\,dx < \eta \int_\mu^\nu \frac{L}{x^{1+\rho}\alpha^\rho}\,dx \leqq \eta \int_a^\nu \frac{L}{x^{1+\rho}\alpha^\rho}\,dx < \eta \frac{L}{\rho c^\rho},$$

which combined with (70) gives

$$\left| \int_\mu^\nu f(x)\phi(\alpha, x)\,dx \right| < |\phi(\alpha, \nu) \int_\mu^\nu f(x)\,dx| + \eta \frac{L}{\rho c^\rho}$$

$$< \eta \left[K + \frac{L}{\rho c^\rho} \right] = \eta \frac{L + K\rho c^\rho}{\rho c^\rho} < \delta.$$

Hence in all three cases we have established the inequality (68), q being taken equal to m.

For $\alpha = 0$ the integral (63) becomes the integral (62), and since the latter is convergent, we can choose m_1 such that

$$\left| \int_\mu^\infty f(x)\phi(0, x)\,dx \right| = \left| \int_\mu^\infty f(x)\,dx \right| < \delta \qquad (\nu > \mu \geqq m_1).$$

To establish the inequality (68) we have now only to take q equal to the greater of the two quantities m and m_1.

Theorem V has been proved under the assumption that a is greater than zero. As in the treatment of summable integrals, however, the theorem can easily be extended to the case where a is any constant. We shall show this by proving the following corollary:

COROLLARY. *If, in Theorem* V, *a is any constant and the convergence factor $\phi(\alpha, x)$ satisfies conditions (a), (b), (c), (d), (e), and (f) for $x \geqq b$, where b is any positive constant, and conditions (a) and (d) for $a \leqq x \leqq b$, the theorem holds without change.*

Let

$$\int_a^b f(x)\,dx = B.$$

We have

$$A = \int_a^\infty f(x)\,dx = \int_a^b f(x)\,dx + \int_b^\infty f(x)\,dx$$

and consequently

$$\int_b^\infty f(x)\,dx = A - B.$$

Hence by Theorem V

$$\int_b^\infty f(x)\phi(\alpha, x)\,dx = G(\alpha)$$

is absolutely and uniformly convergent for all values of $\alpha \geqq 0$ and therefore defines a continuous function for such values.

Then

$$\int_a^b f(x)\phi(\alpha, x)\,dx + \int_b^\infty f(x)\phi(\alpha, x)\,dx = \int_a^\infty f(x)\phi(\alpha, x)\,dx$$

is absolutely and uniformly convergent for all values of $\alpha \geqq 0$ and therefore defines a continuous function for such values.

The convergence factor $e^{-x\alpha}$ satisfies the conditions of Theorem V. Hence if the integral

$$\int_a^\infty e^{kx} f(x)\,dx$$

is convergent, the integral

$$\int_a^\infty e^{(k-\alpha)x} f(x)\,dx$$

converges uniformly in the interval $\alpha \geqq 0$. If now we set

$$e^{(k-\alpha)} = m, \qquad e^k = m_1,$$